# A MATHEMATICAL STUDY ON SOME MHD FLUID FLOW PROBLEMS

**SYNOPSIS SUBMITTED TO MADURAI KAMARAJ UNIVERSITY IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF**

**DOCTOR OF PHILOSOPHY IN MATHEMATICS**


***Submitted by***

**T.NITHYA**

**(Reg. No. P5189)**

***Research Supervisor***

**DR. V. K. SANTHI, M.Sc., M.Phil., Ph.D.,**
**Associate Professor**
**PG Department of Mathematics**
**Sri Meenakshi Government Arts College for Women (Autonomous),**
**Madurai-625002**

***Co-Guide***

**DR. V. ANANTHASWAMY, M.Sc., M.Phil., Ph.D.,**
**Assistant Professor**
**Research Centre and PG Department of Mathematics**
**The Madura College (Autonomous)**
**Madurai – 625011**



**DEPARTMENT OF MATHEMATICS**
**MADURAI KAMARAJ UNIVERSITY**
**(University with potential for excellence)**
**MADURAI – 625 021**
**TAMIL NADU, INDIA**
**MAY 2021**

# INTRODUCTION

## 1.1 Mathematical modelling

Mathematical modelling is a process that utilizes math to represent, analyze, make likelihood or otherwise provide insight into real-world occurrences. In some system complex relationships cannot be represented physically or the physical representation may be difficult and take time to construct. For that reason a more abstract model is used with the assistance of symbols. These are general rather than explicit and can describe varied situation. Besides they can be employed easily for purposes of experimentation and prediction. When the concept of a model is stretched to the area of mathematics, it is beneficial to know in a quantitative sense how important or how appropriate the variables are in a model with respect to their impact on the solution. The mathematical models represent clear relationships and interrelationships among the variables and other factors considered important in solving problems.

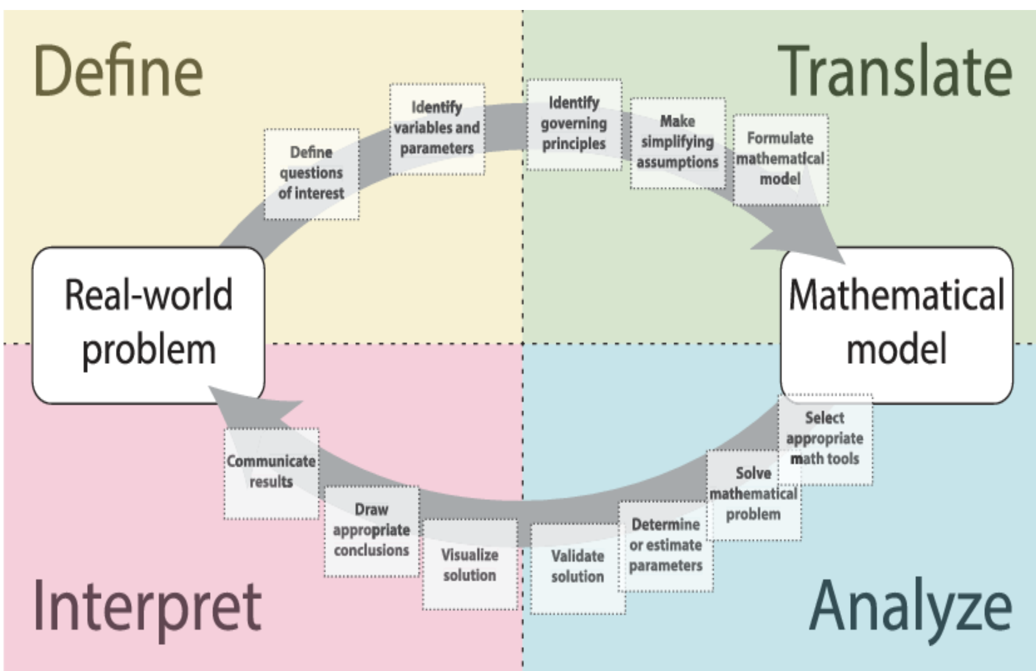


**Fig:1.1 Schematic representation of mathematical modeling**

These Models are wide-ranging rather than specific and can designate diverse situations. Also they can be handled easily for purposes of experimentation and prediction. Mathematical modelinghas numerous applications in different fields:

In Electrical engineering, mathematical modelingand simulations are important during the designing stage and the operational stage of electrical power systems. These electrical power systems are typically large, complex and are spread over a wider geographical area and are frequently made of many different electrical devices. Mathematical models are also used when launching the stability of electrical circuits, when investigating microchips and during the enhancement of power supply networks. The Circuit Differential equations (Kirchhoff laws) are usually used in describing electrical circuits. In mechanical engineering, mathematical modelling is of prominence in crash simulations[ 1 ] and instructuraloptimization of motor vehicle skeleton under different amount of load / stress. In civil and structural engineering, mathematical models are of significance in designing buildings capable of repelling earth quakes to assure safety and security of building occupants and assets [2]. In the field of Agriculture, mathematical models are of importance in the following areas: in modelling of agricultural production systems [3] and [4], formatting the amount of feed and energy supply required by ruminant animals which will reduces wastage and improves efficiency when it comes to production. In meteorological science, mathematical modellingis important in areas such as: modellingof happening and progression of tornadoes [5], modelling of evolution and occurrence of hurricanes [6]. These models can be of use in prevention of major disasters from happening as a result of timely forecast of occurrence of these natural disasters. Rescue measures such as evacuation of people to safer places, protective of power outages due to destruction of electrical transmission lines and acquiring of medical supplies can be planned for in advance. In the field of medicine, mathematical modellingin areasuch as progression of tumourcells [7] is of great importance. Other mathematical models in medicine are models on osteoarthritis that are used to judge the articular cartilage function and possible failure sites in joints. Mathematical models and equations leading mechanics and fluid flow have also been used to explain the flow of synovial fluid during the displacement of bone element and cartilage when subjected to various stress configurations.

## 1.2 Boundary value problems

A boundary value problem is a given differential equation comprising of the finding of a solution to the given differential equation satisfying a set of conditions defined at the boundary points. A boundary condition is the authorization of values of the unknown solution and its derivatives at more than one point. Boundary value problems are of great importance as many of the biological, physical science and chemical science problems can be converted into a boundary value problem. In particular, MHD fluid flow problems involves boundary layer flow. On that ground, MHD problems can be simulated and modeled as boundary value problems. The behavior of the flow and the characteristic of the fluid can be investigated and studied by solving these boundary value problems.

## 1.3 Non-linear process

Non-linearity is a term used to designate a situation where there is not a straight-line or direct relationship between an independent and a dependent variable. In a nonlinear relationship, changes in the output do not change in direct proportion to changes in any of the inputs.

A linear relationship generates a straight line when plotted on a graph, whereas a non-linear relationship does not generate a straight line but instead generates a curve. Non-linearity scrutinizes the cause and effect relationships. Non-linear phenomena performs in a extensive variety of scientific applications such as plasma physics, solid state physics, optical fibers, biology, fluid dynamics and chemical kinetics. Thermal convection in a fluid, Taylor vortex flow, turbulence, the production of coherent light by laser, chemical oscillations offer some well-established examples of this abundant property of a non-linear systems. In order to accelerate fluid flow, heat transfer, and other correlated physical phenomena, it is essential to describe the allied principles in mathematical terms. Nearly all the physical phenomena of interest are obtained by principles of conservation and conveyed in terms of non-linear partial or ordinary differential equations stating these principles. These complicated non-linear differential equations can be solved numerically in many cases; on the other hand, analytical solutions for fluid flow and heat transfer problem can still play a very significant role in science and engineering, yet in the current digital era. This is because analytical solutions have the big benefit of exposing directly the parameters which impact the solution.

## 1.4 Magnetohydrodynamics(MHD) flow

Hydromagnetic(MHD) was first found by Hannes Alfvén(1908-1995). MHD as a field deals with dynamics of conductive fluids in magnetic fields. These conductive fluids include Liquid metals (gallium, mercury and molten iron), plasmas (such as solar atmosphere) and strong electrolytes. In MHD, when the magnetic field and the conducting fluid comes into interaction, electric current of density $j$is induced into the conducting fluid which results in induced magnetic field. The total field $B$ interacts with induced current developing Lorentz force $F = j \, X \, B$. MHDhas a wide-ranging application and they include: MHDpump, MHDpropulsion, MHD generators and MHDflow meters.

In MHD pump, the magnetic field, electric field and the axis of the duct carrying the conductive liquid are mutually perpendicular to each other. The conductive liquid licenses the flow of electric current in the duct. The pumping action is performed by the Lorentz force arising from the interaction of magnetic field and electric field. Since MHD pumps have no movable parts the chance of mechanical failure is reduced. MHD pump has its prominent application following areas:MHDmicro pumps are used as micro syringes for diabetics. In fusion research, MHD pumps are used to create high impact velocities and in cooling of nuclear reactors by pumping sodium coolant in the reactor core.

The MHD generator has the ability to work under very high temperatures than thetraditionalelectric generators. The thermalor kineticenergy directly is directly converted into electrical energy. As MHD generators have no movable parts, the chances of mechanical failure gets reduced. The simplest MHDgenerator has a gas nozzle that serves as a combustion chamber injecting pulses of the gas into the duct. The walls of the duct act as electrodes. The first MHD generator was developed with copper disks and horse shoe magnet in 1831byMichael Faraday. The powerful electromagnet in Faraday's generator actsas a source of magnetic field andthe current flowing between the two installed electrodes which are perpendicular to magnetic field serves as the main electrical output of the MHD generator.

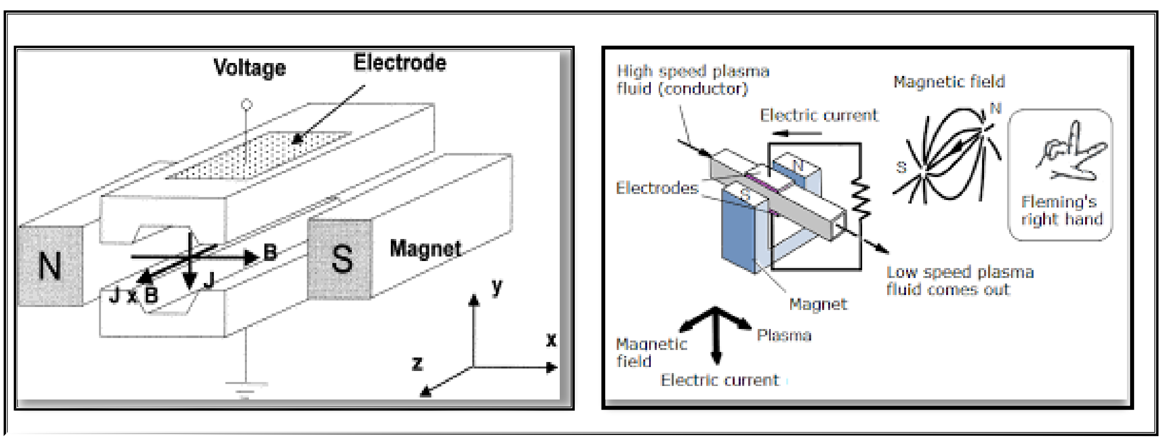


**Fig: 1.1 MHD micro pump and MHD generator**

The MHD propulsion is a substitute to mechanical propellers in propelling marine vessels such as military submarines. It disables the problem of cavitation and is related with the movement of propellers which is a benefit in military where stealth is necessary [8]. MHD propulsion is attained when the sea water is drawn into MHD pumps within the submarine, where the magnetic field and electric current directed through the sea water relates giving rise to Lorentz force. The Lorentz force, energies water out of the vessel making the vessel to accelerate in the opposite direction. WithMHD propulsion greater speeds of the sea water vehicles can be achieved provided large magnets of high magnetic field strength are built for the marine vessel thruster duct [9].

The MHDflow meter's is modelled based on Faraday's law of electro magnetic induction. In MHDflow meters, a magnetic field is generated through the conductive fluid and passed inside the pipe which leads to production of voltage in the fluid as per the Faraday's law. The voltage generated in the fluid is directly proportional to the velocity of the flowing fluid. MHDflow meters can also be used to determine the rate of blood flow through blood vessels. The first use of MHD blood flow meters was by Kolin(1936). MHD blood flow meters are used in surgery to determine the amount of blood flowing through a vessel before, during and after the surgery [11].

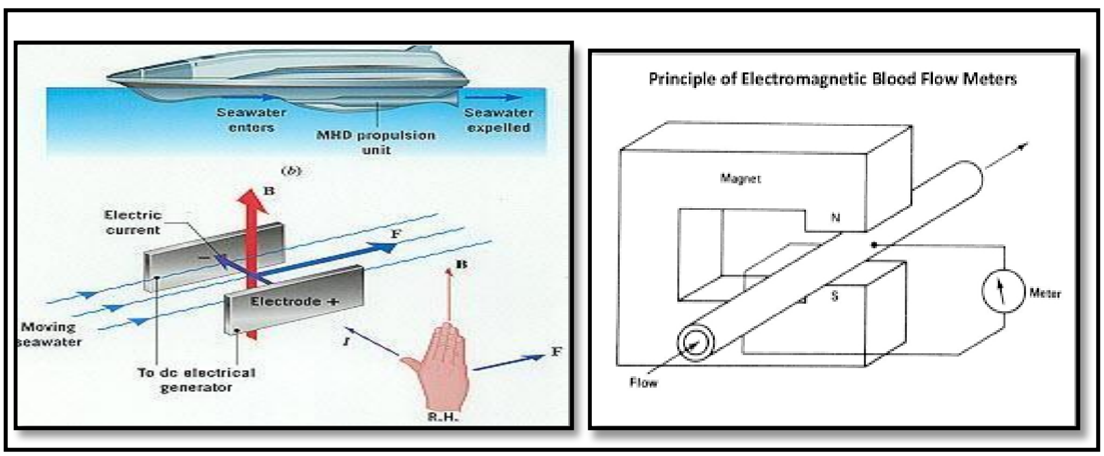


**Fig: 1.2 MHD Propulsion and MHDBlood Flow meters**

## 1.5 Boundary layer flow

The idea of boundary layer in a fluid flow was first commenced by Ludwig Prandtlin1904. Boundary layer flow has major role in fluid dynamics. It is of importance in determining friction drag of bodies moving in fluids such as viscous drag on aerodynamics (airplanes, rockets, and projectiles such as missiles), hydrodynamics (ships, submarines and torpedoes), automobiles (motor vehicles) and engineering structures such as buildings and bridges.

A boundary layer in fluid flow can be described as a thin layer of viscous fluid just neighbouring thesurface or the wall in which the fluid has a zero velocity at the wall / plate and a free stream velocity $u_0$ far away from plate. The fluid above the surface of the plate is moving with shearing happening between its layers. The shear stress happening between the surface of theplate and the first moving layer of the fluid adjacent to the plate is known as the wall shear stress $T_W$. The boundary layer thickness $\delta$ is represented as a function of the Reynolds number and it refers to the distance between the solid wall andthe height above the surface of the wall where the velocity of the fluid is 99% of the free stream velocity $u_0$. In boundary layer flow, the hydrodynamic and thermalboundary layers are of great significance. In hydrodynamic boundary layer, the fluid velocity is zero at the plate and its value increases to a free stream value $u_0$ far away from the plate. In thermal boundary layer, the temperature of the fluid varies from the wall temperature $T_0$ to the free stream value $T_\infty$ far away from the wall. The fluid particles in contact with the solid wall acquire temperature equal to that of the wall. If the wall temperature is higher compared to the rest of the fluid, the fluid particles in contact with the wall exchanges heat with those in the neighboring layers leading to the development of a thermal gradient in the fluid. The understanding of hydrodynamicand thermal boundary layer is of significance in fluid mechanics since velocity is an important component in mass, momentum and energy equations while temperature gradient in the thermal boundary layer influences heat transfer in the fluid.

## 1.6 Equations governing fluid dynamics

The equations of fluid flow (fluid dynamics) consist of three important equations namely:

(i) The Equation of continuity which is derived from conservation of mass for a system (i.e., mass can neither be created nor destroyed).

(ii) Navier-Stokes (Momentum) Equation which is derived from Newton's second lawof motion $\left(\vec{F} = m\vec{a} = m\frac{d\vec{u}}{dt}\right)$

(iii) The energy equation which is derived from the first law of thermodynamics (i.e., the amount of heat added to the system $dQ$ is equal to change in internal energy $dE$ plus the amount of energy lost due to work done on the system $dW$ that is $dQ = dE + dW$ )

### 1.6.1 The continuity equation

To derive the continuity equation, we apply the mass conservation principle on an infinitesimal volume of fluid element within a moving fluid. The equation of continuity for differential element in the Cartesian coordinate system is of the form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad or \quad \frac{\partial \rho}{\partial t} + \nabla.\left(\rho\vec{V}\right) = 0 \qquad (1)$$

where, $\rho$ is the density of the fluid, $(u, v, w)$ is velocity component of the fluid in $(x, y, z)$ directions respectively ,$\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ and $\nabla.(\vec{V})$ refers to divergence of the velocity vector. Equation (1) is the Continuity equation for a compressible fluid in a rectangular cartesiancoordinate system.

If the flow of the fluid is steady (i.e., density is not a function of time), $\frac{\partial \rho}{\partial t} = 0$ and equation (1) simplifies to:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad or \qquad \nabla.(\rho\vec{V}) = 0 \quad (2)$$

For incompressible flow (density is constant), the material derivative of density is zero i.e., $\frac{\partial \rho}{\partial t} = 0$ and the continuity equation for incompressible flow becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad or \qquad \nabla.(\vec{V})=0 \qquad (3)$$

### 1.6.2 Momentum (Navier-Stokes) equation

The momentum equation for a flowing fluid is of the form

$$\frac{\partial u}{\partial t}+(V.\nabla)V=\frac{1}{\rho}\left[-\nabla P+\mu\nabla^2 V\right]+F \qquad (4)$$

where, F represents forces acting on flowing fluid.

If forces acting on flowing fluid are steady, the thermal expansion and the Lorentz force are created by the Magnetic field , and so the Navier-Stokes equation (4) takes the form:

$$\frac{\partial u}{\partial t}+(V.\nabla)V=\frac{1}{\rho}\left[-\nabla P+\mu\nabla^2 V\right]+\rho\beta g\Delta T+\frac{1}{\rho}j\,\mathrm{X}\,B \qquad (5)$$

where, V denotes fluid's velocity, $\rho$ represents density of the fluid, P denotes pressure, $\mu$ is the dynamic viscosity, g stands for gravitational force, $\beta$ stands for magnetic field, $j$ is the electric current and B denotes the magnetic field.

### 1.6.3 Energy equation

The energy equation which is obtained from the first law of thermodynamics takes the form:

$$\rho C_P\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+q^*+\mu\phi \qquad (6)$$

where,

$$\phi=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right]+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2 \qquad (7)$$

$\rho C_P$ is the heat capacitance of the fluid, $(u,v,w)$ is the velocity component of the fluid in the $(x,y,z)$ directions respectively, $T$ refers to local temperature of the fluid, $k$ is the thermal conductivity of the fluid and $q^*$ is the heat flux.

### 1.6.4 Equations governing MHD fluid flow

If the flowing fluid happens to be in a magnetic field, then the equations governing such a flow are Navier-Stokes (momentum) equation and Maxwell's equations of electro magnetism. The Maxwell's equations of electro magnetism are:

$$\nabla \mathrm{X} B = \mu_0 j \quad (8)$$

$$\frac{\partial B}{\partial t} = -\nabla \mathrm{X} E \quad (9)$$

$$j = \sigma(E + V \mathrm{X} B) \qquad (10)$$

Equations (8), (9) and (10) are the Ampere's law, theFaraday's law and the Ohm's law respectively. Here, $\mu_0$ is magnetic permeability, $\sigma$ is the electrical conductivity of the fluid, $j$ represents electric current density, $E$ stands for electric field and $B$ is the magnetic field.

These differential equations governing the flow should be solved simultaneously, either analytically or numerically.

## 1.7 Techniques of solving non-linear differential equations

In Mathematics, some problems can be solved analytically or numerically. An analytical or closed-form solution provides a good insight in phenomena under interest. Whereas, Numerical methods give a clue what kind of closed-form solution could be achieved. In this thesis, methods like Homotopy analysis method, modified homotopy analysis method and q-homotopy analysis method are adapted to obtain a semi-analytic solution for the system of strongly non-linear differential equations that governs MHD fluid flow.

### 1.7.1 Basic perception of Homotopy analysis method

Homotopy analysis method (HAM) is an efficient tool for handling various kind of functional equations like PDEs, ODEs, BVPs, system of equations, fractional forms, integral equations and many other families. It is a technique mainly focused for non-linear equations. Other than applied mathematics, the method is enrouted into different fields in engineering and sciences, even the chemists, physicians, biologists also take advantage of this powerful analytic technique for coping their own problems and equations. It is a method which is well-established from a mathematical point of view, Liao and his colleagues have successfully tried to enrich the method. Many others have tried to find its relation with classic methods and give a general algorithm for its implementation [24]-[33].

In the framework of the HAM, if the equation under study has a solution as a series expression, then HAM is able to give a good approximation of this solution. HAM is equipped with convergence control parameter and the final representation of the solution is dependent upon this parameter. Using this parameter one can easily control and possibly extend the convergence region of the solution obtained. This advantage enables it as a favourite technique for applied mathematician.

It has been proved that the HAM solution with this CCP is a Taylor series expansion of the exact solution at some point. The taylor series expansions usually have restricted convergence regions. The CCP is a gift of HAM to the community of analytic and semi-analytic methods which can easily overcome this lack.

Consider the differential equation,

$$N[u(\tau)] = 0 \quad (11)$$

where, $N$ is a non-linear operator, $\tau$ is the independent variable, $u(\tau)$ is an unknown function.

By Means of the Homotopy theory, we construct a Homotopy

$$(1-q)L[\varphi(\tau;q) - u_0(\tau)] - qhH(\tau)N[\varphi(\tau;q)] = 0 \qquad (12)$$

where $q \in [0,1]$ is called the embedding parameter, $h$ is an auxiliary parameter called convergence control parameter such that $h \in [-1,1]\, and\, h \neq 0$, $L$ is an auxiliary linear operator, $H(\tau) \neq 0$ is an auxiliary function, $u_0(\tau)$ is an initial approximation of the solution $u(\tau)$ and $\varphi(\tau;q)$ is the unknown solution.

Eqn. (12) is called the zeroth-order deformation equation. The solution varies from the initial solution to the solution.

When $q = 0$ *and* $q = 1$,

$\varphi(\tau;0) = u_0(\tau)$ ( the initial approximation of the solution ) and

$\varphi(\tau;1) = u(\tau)$ (the desired solution) respectively.

On expanding $\varphi(\tau;q)$ in Taylor series with respect to q,

$$\varphi(\tau,q)=u_0(\tau)+\sum_{m=1}^{\infty}u_m(\tau) \qquad (13)$$

where,

$$u_m(\tau)=\frac{1}{m!}\frac{\partial^m\varphi(\tau,q)}{\partial q^m}\bigg| q=0 \quad (14)$$

The convergence of the series (13) depends upon the auxiliary parameter *h*. If it is convergent at *q=1*, then

$$u(\tau)=u_0(\tau)+\sum_{m=1}^{\infty}u_m(\tau) \qquad (15)$$

Differentiating the zero order deformation equation m times with respect to *q* and then dividing them by *m!*and finally setting *q=0* we get the following $m^{th}$ order deformation.

$$L(u_m(\tau)-X_m u_{m-1}(\tau)=hR_m(u_{m-1}) \qquad (16)$$

where,

$$R_m(u_{m-1})=\frac{1}{m!}\frac{\partial^{m-1}N[\varphi(\tau:q)]}{\partial q^{m-1}}\bigg| q=0 \quad (17)$$

It is emphasized that $u_m(\tau,q)$ for $m\geq 1$is governed by the linear equation (15) with linear boundary conditions that comes from the original problem, which can be solved. If the equation does not possess a unique solution, the HAM will give a solution among many other possible solution.

Remarkable features of HAM:

1. It is independent of any small / large physical parameters. The HAM can be applied to solve most of nonlinear problems in science, finance and engineering, especially those without small / large physical parameters.

2. It provides a great freedom and large flexibility to chose the auxiliary linear operator and base functions. Using such kind of freedom, some complicated non-linear problems can be solved in a much easier way.
3. It provides a convenient way to guarantee the convergence of solution series so that it is valid for highly nonlinear problems.
4. The convergence control parameter is able to control and extend the convergence region of the series solution.
5. The special case of HAM is the well-known Homotopyperbutation method.

### 1.7.2 Basic perception of Modified Homotopy analysis method

Modified Homotopy Analysis method is a technique of solving non-linear boundary value problems defined at infinity [34]-[37] . This method proposes a procedure of obtaining series solutions that contain exponential and other periodic functions.

Consider the differential equation

$$N[u(\tau)] = 0 \quad (18)$$

where , $N$ is a non-linear operator, $\tau$ is the independent variable, $u(\tau)$ is an unknown function.

By Means of the Homotopy theory, we construct a Homotopy

$$(1-q)L[\varphi(\tau;q) - u_0(\tau)] - qhH(\tau)N[\varphi(\tau;q)] = 0 \qquad (19)$$

where $q \in [0,1]$ is called the embedding parameter, $h$ is an auxillary parameter called convergence control parameter such that $h \in [-1,1]\, and\, h \neq 0$, $L$ is an auxillary linear operator, $H(\tau) \neq 0$ is an auxillary function, $u_0(\tau)$ is an initial approximation of the solution $u(\tau)$ and $\varphi(\tau;q)$ is the unknown solution.

There are some rules for the choice of the solution-expression of a given equation $N[u] = 0$

1. Equations defined in a finite domain:
    - Polynomials and Fourier series can be always used as the solution-expression.

- Chebyshev series gives the best approximation for arbitrary continuous solutions of $N[u]=0$.

2. Equations defined in an infinite domain:
   - Periodic base functions should be used, if the solution is periodic and Non-periodic, if the solution is not periodic.
   - The solution-expression should satisfy as many asymptotic properties of solution as possible, if the solution is not periodic.

The above rules are not absolute, especially when the unknown solution is non-periodic and defined in an infinite domain. Even a unique solution of $N[u]=0$ can be often expressed by different types of base functions. So, if a little information about the solution-expression is known, one can always guess some form of its solution-expression and check whether these guesses are correct or not. Thus, solution-expression of non-linear differential equations contain the exponential functions which exponentially tend to zero at infinity.

The idea of the so-called solution-expression plays a vital role in the frame of HAM, because the start-point is to choose the appropriate initial approximation $u_0$ and the corresponding auxiliary linear operator *L*. Modified HAM provide as a convenient way of choosing the initial approximation and the corresponding linear operator.

Assume that a solution expression is chosen for a given equation $N[u]=0$. Our aim is to find a proper initial approximation $u_0$ and a proper auxiliary linear operator *L* such that the corresponding homotopy-series converge. Obviously, the initial approximation $u_0$ must obey the so-called solution expression. Then the initial approximation can be chosen by:

$$u_0(x,t) \approx \sum_{k=1}^{n_0} \bar{a}_k e_k(x,t) \qquad (20)$$

where, $e_k(x,t)$ is the base function, $\bar{a}_k$ is unknown constant and $n_0$ is equal to or greater than the number of linear boundary / initial conditions, denoted by k. Then, enforcing the above initial approximation to satisfy the $k$ boundary / initial linear conditions, we have $n_0 - k$ unknown coefficients left. When $n_0 = k,$ then all coefficients of the initial approximation are known so that it is completely determined. However, when $n_1 = n_0 - k > 0,$ we have $n_1$ unknown coefficients, denoted by $b_1, b_2, ......b_{n_1}$. To gain an optimal initial approximation, we defined the squared residual of the governing equations:

$$E_0(b_1, b_2, ..., b_{n_1}) = \int_{\Omega} [N(u_0)]^2 d\Omega \quad (21)$$

It is well-known that the minimum of the squared residual $E_0(b_1, b_2, ...., b_{n_1})$ is determined by a set of nonlinear algebraic equations.

$$\frac{\partial E_0}{\partial b_k} = 0, \qquad 1 \le k \le n_1, \qquad (22)$$

whose solution gives the optimal values of the unknown coefficients. In this way, we obtain an approximation $u_0$. Therefore in the modified HAM, when the domain is inifinite, the initial guess of the solution-expression and the corresponding auxiliary linear operator $L$ is determined by the initial / boundary conditions. By knowing the auxiliary linear operator and the initial approximation of the solution, the homotopy can be constructed as in the traditional HAM and the successive approximation can be obtained.

Remarkable features of modified HAM:

1. It gets all the advantages of the traditional HAM like convergence control parameter.
2. It provides a simpler method of choosing the auxiliary linear operator and the initial approximation of the series solution satisfying the boundary conditions at infinity.

### 1.7.3 Basic perception of q-Homotopy analysis method

Consider the non-linear differential equation

$$N[u(x,t)] - f(x,t) = 0 \qquad (23)$$

where, $N$ is a nonlinear operator, $(x,t)$ denotes the independent variables, $f(x,t)$ is a known function and $u(x,t)$ is an unknown function.

By q-Homotopy analysis method [38]-[40], we construct the zero order deformation equation as:

$$(1-nq)L[\phi(x,t;q)-u_0(x,t)] = qhH(x,t)(N[\phi(x,t;q)]-f(x,t)) \qquad (24)$$

where, $n \geq 1$, $q \in \left[0, \frac{1}{n}\right]$ denotes the embedding parameter, $L$ is an auxiliary linear operator with the property $L[f]=0$ when $f=0, h \neq 0$ is an auxiliary parameter, $H(x,t)$ denotes a non-zero auxiliary function.

when $q=0$ and $q=1/n$, $\phi(x,t,0)=u_0(x,t)$ *and* $\phi\left(x,t,\frac{1}{n}\right)=u(x,t)$ respectively.

Thus as $q$ increases from 0 to $\frac{1}{n}$, the solution $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Having the freedom to choose $u_0(x,t)$, $L$ ,$h$, $H(x,t,)$, one can choose them appropriately, so that the solution $\phi(x,t;q)$ of (12) exists for $q \in \left[0, \frac{1}{n}\right]$.

Expanding $\phi(x,t;q)$ in Taylor's series, we get :

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m \quad (25)$$

where, $u_m(t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m} \bigg| q=0$ (26)

Assume that $h, H(x,t), u_0(x,t), L$ are properly chosen such that the series (12) converges at $q=\frac{1}{n}$ and :

$$u(x,t) = \phi\left(x,t;\frac{1}{n}\right) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\left(\frac{1}{n}\right)^m \qquad (27)$$

Defining the vector $u_r(x,t) = \{u_0(x,t), u_1(x,t), u_2(x,t), .....u_r(x,t)\}$. Differentiating (24) $m$ times with respect to $q$ and then setting $q=0$ and finally dividing them by $m!$ , we have the $m^{th}$ order deformation equation:

$$L[u_m(x,t) - k_m u_{m-1}(x,t)] = hH(x,t)R_m(u_{m-1}(x,t)) \quad (28)$$

where,

$$R_m(u_{m-1}(t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\left(N[\phi(x,t;q)] - f(x,t)\right)}{\partial q^{m-1}}\bigg| q = 0 \qquad (29)$$

and

$$K_m = \begin{cases} 0 & m \leq 1 \\ n & otherwise \end{cases}$$

It should be emphasized that $u_m(x,t)$ for $m \geq 1$ is governed by the linear equation (28) with the boundary conditions that come from the original problem. Due to the presence of the factor $\left(\frac{1}{n}\right)^m$, more chances for convergences may occur or even much faster convergence can be obtained better than the standard HAM. It should be noted that the cases of $n$=1 in Eqn.(12), standard HAM can be reached.

Remarkable features of modified q- HAM:

1. It exhibits all the benefits of the traditional HAM.
2. Along with the convergence control parameter h, q-Homotopy analysis method offers the factor n, so that the convergence can be obtained much faster than the traditional HAM.
3. When the fraction factor $q$ gradually increases continuously toward 1/n, the system goes through a sequence of deformations, and the solution at each stage is close to that at the previous stage of the deformation.

## 1.8 Objectives ofthestudy

The main objective of the present study are:

- To examine the behaviour of plain PoiseuilleMHD flow of an electrically conducting fluid when subjected to thermal conductivity and magnetic field by solving the coupled, non-linear differential equations governing the illustration analytically using Homotopy analysis method (HAM).

- To obtain the semi-analytic solution for the velocity and temperature of the boundary layer flow of incompressible viscous fluid by solving the governing differential equations analytically using Modified Homotopy Analysis Method.
- To solve the Navier-Stokes equations for a steady magneto hydrodynamic (MHD) flow between two parallel porous plates by adapting the $q$-Homotopy analysis method to solve the non-linear differential equation that influence the flow.
- To investigate the MHD flow by solving a system of highly non-linear differential equation governing MHD boundary layer flow over a moving vertical porous plate.
- To study the radiation and thermal diffusion effect on a steady MHD free convection heat and mass transfer flow past an inclined stretching sheet with Hall current and heat generation by solving the non-linear differential equations governing the model using Modified Homotopy analysis method.
- To determine the semi-analytic solution for the nonlinear differential equations governing the inherent irreversibility in a steady hydromagnetic permeable channel flow of a conducting fluid with variable electrical conductivity and asymmetric Navier slip at the channel walls in the presence of induced electrical field by exercising Homotopy analysis method.

## 1.9 Organization of the research work

The present work investigates the characteristics of MHD fluid flow problems by solving analytically the governing nonlinear differential equations using suitable asymptotic methods. The obtained results are represented graphically and successfully compared with the previous works.

Chapter 1 contains introduction to mathematical modeling, non-linear phenomena, basic concepts in MHD flow, governing equations in MHD flow, various methods of solving non-linear differential equations, some ideas and methods which are needed for the subsequent chapters.

Chapter 2 examines the study of MHD plane Poiseuille flow in a porous channel with non-uniform plate temperature by solving governing equations analytically using Homotopy analysis method.

Chapter 3 provides the approximate analytical expressions of a boundary layer flow of a viscous fluid using the Modified Homotopy analysis method.

Chapter 4 deals with the study of the Navier-Stokes equations for a steady Magnetohydrodynamic flow by utilizing $q$-Homotopy analysis method.

Chapter 5 is the application of q-Homotopy analysis method in solving a system of highly non-linear differential equation governing the heat and mass transfer on MHD boundary layer flow. Chapter 6 investigates the radiation and thermal diffusion effect on a steady MHD free convection heat and mass transfer flow past an inclined stretching sheet by solving analytically the governing non-linear differential equations using Modified Homotopy analysis method.

Chapter 7 gives the study of the inherent irreversibility in a steady hydromagnetic permeable channel flow of a conducting fluid with variable electrical conductivity by exercising Homotopy Analysis Method to solve the governing equations.

Chapter 8 discusses with overall conclusion and future enhancement of the thesis.

## 1.10 Significanceofthe study

ProblemsinvolvingMHDboundarylayerflowofafluidofvaryingviscosity subjectto thermal radiation andNewtonianheatingareofgreatimportancetoengineeringand industrial applications due to their vast applications in thermalinsulation, heat exchangers,geothermalreservoir,coolingofnuclearreactors,enhancedoilrecovery, solarenergycollection,designingof coolingsystemsforelectronicdevices,packed-bed catalyticreactorsetc.Heattransferby thermal radiationisalsoofgreatsignificanceto engineeringprocessesoccurringathightemperaturesanditisimportantinthedesigning of equipmentinNuclearpowerplants,gas turbines,andpropulsiondevicesforaircraft, missiles,satellites andspacevehicles.

The search of explicit solutions to the non-linear equations representing the flow model is one of the principal objectives in MHD flow problems. Some Numerical methods such as Shooting method, Rungekutta and many other methods have been extensively used to solve these system of non-linear differential equation [12] - [23]. Practically, there is no unified method that could be used to handle all types of nonlinear problems.In this thesis, methods like Homotopy analysis method, q-homotopy analysis method and Modified homotopy analysis method are adapted to solve the system of strongly non-linear differential equations that governs the MHD fluid flow.

**1.10 REFERENCES**


[1] T. Omar, A.Eskandarian, & N. Bedewi, Vehicle crash modelling using recurrentneuralnetworks, MathematicalandComputer Modelling,Vol.28,31–42, (1998).

[2] R. Rahgozar,A.R. Ahmadi,Y.Sharifi, A simplemathematicalmodelfor approximateanalysis of tall buildings,Applied Mathematical Modelling, Vol.34, pp: 2437–2451,(2010).

[3] E. Audsley,Anarablefarmmodeltoevaluatethecommercialviability ofnew machinesortechniques, JournalofAgriculturalEngineeringResearch,Vol. 26,pp:135–149,(1981).

[4] D.K. Lambert, B.A. McCarl, Risk Modeling Using Direct Solutionof NonlinearApproximations oftheUtilityFunction,AmericanJournalofAgricultural Economics,Vol. 67,pp: 846 -852(1985).

[5] N. Dotzek,J. Grieser,H.E. Brooks,Statisticalmodelingof tornadointensity distributions, Atmospheric Research,Vol. 67, pp: 163–187, (2003).

[6] M.A. Bender,I.Ginis,R.Tuleya,B.Thomas,T. Marchok,T,TheOperational GFDLCoupledHurricane–OceanPredictionSystem andaSummaryofIts Performance,MonthlyWeather Review,Vol. 135,pp: 3965–3989, (2007).

[7] Y. Kim,M.A.Stolarska,H.G. Othmer,A hybridmodelfortumorspheroid growthinvitroI:Theoretical developmentandearlyresults, MathematicalModels andMethods inAppliedSciences,Vol. 17(supp01), pp: 1773–1798, (2007).

[8] Bednarczyk, A. Adalbert,Nuclear ElectricMagnetohydrodynamicPropulsionforSubmarine, Calhoun: The NPS Institutional

Arcive, Monterey, California, USA, (1989).

[9] D.L. Mitchell,D.U.Gubser,Magnetohydrodynamicshippropulsionwith superconductingmagnets, JournalofSuperconductivity,Vol. 14, pp: 349–364, (1988).

[10] R.M. Khristinich,V.N. Timofeyev,V.V. Stafievskaya,A.V.Velenteyenko,MoltenMetal ElectromagneticStirringin Metallurgy,International Scientific Colloquim,Hannover,29–36, (2003).

[11] D.G. Wyatt, Theelectromagneticblood flowmeter, Journalof PhysicsE: ScientificInstruments,Vol. 1,pp: 1146-1152(1968).

[12] M.A. AbdEl-Naby,E.M.E. Elbarbary,N.Y. AbdElazem,Finitedifference solutionofradiationeffectsonMHDunsteadyfree-convectionflowoververtical porous plate, AppliedMathematics andComputation,Vol. 151(2), pp: 327–346, (2004).

[13] S. Aïboud,S. Saouli,Entropy analysisforviscoelasticmagnetohydrodynamicflowoverastretching surface,International Journal of Non-LinearMechanics,Vol. 45, pp:482–489, (2010).

[14] O.D. Makinde,OnMHDheatandmasstransferoveramovingverticalplate with aconvectivesurfaceboundary condition,TheCanadianJournalofChemical Engineering,Vol. 88, pp. 983–990, (2010).

[15] O.D. Makinde,Thermodynamicsecondlawanalysisfora gravity-driven variableviscosityliquidfilmalonganinclinedheatedplatewithconvectivecooling,JournalofMechanicalScienceandTechnology,Vol. 24, pp. 899–908, (2010).

[16] O.D. Makinde,A. Aziz,Secondlawanalysisforavariableviscosity plane Poiseuilleflowwithasymmetricconvectivecooling, Computers &Mathematics withApplications,Vol. 60,pp: 3012–3019, (2010).

[17] O.D. Makinde,O.A. Beg,Oninherentirreversibilityin areactive hydromagneticchannelflow,JournalofThermalScience,Vol. 19,pp. 72–79, (2010).

[18] E.M. Sparrow, R.D. Cess, Free convection with blowing or suction, Int. J. Heat Mass Transfer, Vol. 83, pp: 243-248, (1961).

[19] I.U. Mbeledogu, A.R.C. Amakiri, A. Ogulu,Unsteady MHDfree convectiveflowofacompressiblefluidpastamovingverticalplateinthepresence ofradiativeheattransfer,InternationalJournalofHeatandMassTransfer,Vol.50,pp. 1668–1674, (2007).

[20] S. Ahamed, N. Ahamed, Two-dimensional MHD oscillatory flow along a uniformly moving infinite vertical porous plate bounded by porous medium, Indian .J. Pure Appl. Math, Vol.35, pp: 1309-1319, (2004).

[21] P. Sreenivasulu, T. Poornima, N. Bhaskar Reddy, Thermal radiation effects on MHD boundary layer slip flow past a permeable exponential stretching sheet in the presence of joule heating and viscous dissipation, Journal of Applied Fluid Mechanics, Vol. 9, pp: 267-278, (2016).

[22] P.M. Kishore, V. Rajesh, S. VijayakumarnVerma, The effectsofthermal radiation andviscous dissipationonMHD heat and mass diffusion flow past an oscillating vertical plateembedded in a porous medium with variable surface condition, Theoret. Appl. Mech., Vol.39, pp: 99-125, (2012).

[23] U. Narusawa,Thesecond-lawanalysisofmixedconvectioninrectangularducts.HeatandMass Transfer, Vol. 37,pp: 197–203, (2001).

[24] S.J. Liao, Y. Tan, A general approach to obtain series solutions of nonlinear differentialequations, Stud. Appl. Math. 119: 297-354, (2007).

[25] S.J. Liao, Beyond Perbutation: Introduction to Homotopy Analysis Method, Chapman &Hall / CRC Press, Boca Raton, (2003).

[26] D.L. Xu, Z.L. Lin, S.J. Liao, M.Stiassnie, On the steady-state fully resonant progressivewaves in water of the finite depth, J. Fluid Mech. 710: 379-418, (2012).

[27] J. Cheng, S.P. Zhu &S.J. Liao, An explicit series approximation to the optimal exerciseboundary of American put options, Commun. Nonlinear Sci. Numer. Simulation, Vol. 15, pp: 1148-1158, (2010).

[28] S. J. Liao, On the analytic solution of magnetohydrodynamic flows of non-

Newtonianfluids over a stretching sheet, Journal of Fluid Mechanics, Vol. 488, pp: 189-212, (2003).

[29] J.H. He, Homotopyperbutation method; a new nonlinear analytic technique, AppliedMathematics and Computation, Vol. 135, pp: 73-79, (2003).

[30] J.H. He, Homotopyperbutation technique, Computer Methods in Applied Mechanics andEngineering Vol. 178, pp: 257-262, (1999).

[31] S. Dinarvand, M.M. Rashidi, H. Shahmohamadi, Analytic approximate solution of threedimensional Navier-Stokes equations of flow between two streatchable disks, Wileyonline library, Wiley periodicals Vol. 26, pp: 1594-1607, (2009).

[32] Z. Ziabakhsh, G. Domairry, Solution of the laminar viscous flow in a semi-porouschannel in presence of a uniform magnetic field by using the homotopy analysis method,Communications in Nonlinear Science Numerical Simulation, Vol. 14, pp: 1284-1294,(2009).

[33] M. Zurigat, S. Momani, A. Alawneh, The multistage HAM: Application to bio-chemicalreaction model of fractional order, International Journal of Computer Mathematics, Vol. 92, pp: 1030-1040, (2013).

[34] Z. Ayati, J. Biazar, B. Gharedaghi, The application of modified homotopy analysismethod for solving linear and non-linear inhomogeneous Klein-Gordan equations, ActaUnversitatisApulensis, Vol.39, pp:31-40, (2014).

[35] D. Ziane, M. HamdiCherif, Modified homotopy analysis method for nonlinear fractionalpartial differential equations, Int. J.Anal.Appl. Vol.14, No.1, pp. 77-87, (2017).

[36] A. Sami Bataineh, M.S.M. Noorani, I.Hashim, Modified homotopy analysis method forsolving systems of second-order BVPs, Vol.14, No.2, pp: 430-442, (2009).

[37] A. Sami Bataineh, M.S.M. Noorani, I.Hashim, On a new reliable modification ofhomotopy analysis method, Vol.14, No.2, pp: 409-423, (2009).

[38] M.A. El-Tawil&S.N. Huseen, The q-Homotopy analysis method (q-HAM), Int. J. of Appl. Math and Mech, Vol. 8, pp: 51-75, (2012).

[39] M.A. El-Tawil&S.N. Huseen, On Convergence of q-Homotopy analysis method,

Int.J.Contemp.Math.Sciences, Vol.8, pp: 481-497, (2013).

[40] S.N. Huseen, Solving the K(2,2) equation bt Means of the q- Homotopy analysis method (q-HAM), International Journal of Innovative Science, Engineering & Technology, Vol.2, pp: 805-817, (2015).

**PROPOSED CONTENT OF THE THESIS**

**CHAPTER 1.** **INTRODUCTION**

**CHAPTER 2.** **A MATHEMATICAL STUDY ON MHD PLANE POISEUILLE FLOW IN A POROUS CHANNEL WITH NON-UNIFORM PLATE TEMPERATURE**

**CHAPTER 3.** **APPROXIMATE ANALYTICAL EXPRESSIONS OF A BOUNDARY LAYER FLOW OF VISCOUS FLUID USING THE MODIFIED HOMOTOPY ANALYSISMETHOD**

**CHAPTER 4.** **MATHEMATICAL ANALYSIS OF THE NAVIER-STOKES EQUATIONS FOR STEADY MAGNETOHYDRODYNAMIC FLOW**

**CHAPTER 5.** **A MATHEMATICAL ANALYSIS OF HEAT AND MASS TRANSFER ON MHD BOUNDARY LAYER FLOW**

**CHAPTER 6.** **A MATHEMATICAL ANALYSIS OF RADIATION AND THERMAL DIFFUSION EFFECT ON A STEADY MHD FREE CONVECTION HEAT AND MASS TRANSFER FLOW PAST AN INCLINED STRETCHING SHEET**

**CHAPTER 7.** **A MATHEMATICAL STUDY ON MAGNETOHYDRODYNAMIC PERMEABLE CHANNEL FLOW**

**CHAPTER 8.** **CONCLUSION AND FUTURE ENHANCEMENT**

## LIST OF PUBLICATIONS BASED ON THE THESIS

1 V. Ananthaswamy, **T. Nithya**, V.K. Santhi, A mathematical study on MHD plane Poiseuille flow in a porous channel with non-uniform plate temperature, Journal of Applied Science and Computations (UGC approved journal-41238), Volume VI, Issue III, pp: 1178-1194, March 2019.

2 V. Ananthaswamy, **T. Nithya**, V.K. Santhi, Approximate analytical expressions of a boundary layer flow of viscous fluid using the Modified Homotopy analysis method, Journal of Information and Computational Science (UGC-CARE List Group-II-Scopus source list) , Volume 9, Issue 8, pp: 534-541, 2019.

3 V. Ananthaswamy, **T. Nithya**, V.K. Santhi, Mathematical analysis of the Navier-Stokes equations for steady Magnetohydrodynamic flow, Journal of Information and Computational Science (UGC-CARE List Group-II-Scopus source list), Volume 10, Issue 3, pp: 989-1003, 2020.

4 **T. Nithya**, V. Ananthaswamy, V.K. Santhi, A mathematical analysis of heat and mass transfer on MHD boundary layer flow, Advances in Mathematics: Scientific Journal (UGC-CARE List Group-II-Scopus source list) 8 No.3, (Special Issue on ICRAPAM), pp: 131-153, 2019.

5 **T. Nithya**, V. Ananthaswamy, V.K. Santhi, A mathematical analysis of radiation and thermal diffusion effect on a steady MHD free convection heat and mass transfer flow past an inclined stretching sheet, Proteus Journal (UGC-CARE List Group-II), Volume 11, Issue 9, pp: 407-443, 2020.

6 V. Ananthaswamy, **T. Nithya**, V.K. Santhi, A mathematical study on magnetohydrodynamic permeable channel flow, Mathematics in Engineering, Science and Aerospace (UGC-CARE List Group-II-Scopus source list), Vol.12,No.1, pp:181-217, 2021.

# LIST OF CONFERENCES / SEMINARS / WORKSHOPS PARTICIPATEDAND PAPER PRESENTED

1 **Presented a paper** titled A mathematical study on magnetohydrodynamic permeable channel flow, in the1st International Conference on Discrete Mathematics and Data Sciences - ICDMDS'18, organized by the School of Humanities and Sciences and the School of Computing held at SASTRA Deemed University, Thanjavur, on 28th and 29th September 2018.

2 **Presented a paper** titled A mathematical analysis of heat and mass transfer on MHD boundary layer flow in the International conference on recent advances in pure and applied Mathematics conducted in Raja Doraisingam Government Arts College, Sivagangai, Tamilnadu, India, on 27th and 28th August, 2019.

3 **Presented a paper** titled Mathematical analysis of the Navier-Stokes equations for steady Magnetohydrodynamic flow, in the3rd International Conference on New Scientific Creations in Engineering and Technology, organized by NadarSaraswathi College of Engineering and Technology, on 19th March 2021.

4 Participated in TNSCST sponsored International conference on Applications of Mathematics in Science and Technology held at AAA college of Engineering and Technology on 25 and 26 July 2019.

5 Participated in One day Workshop on Delay Differential Equations held at VVV college (A), Virudhunagar on 20th September 2019