MARKOV DECISION PROCESS IN SUPPLY CHAIN MANAGEMENT

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By

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CERTIFICATE

This is to certify that the thesis entitled **"MARKOV DECISION PROCESS IN SUPPLY CHAIN MANAGEMENT"** submitted by **Mrs. P.K.Santhi,** Assistant Professor, Department of Mathematics, Jayaraj Annapackiam College for Women , Periyakulam, Theni (District.) for the award of the degree of Doctor of Philosophy in Mathematics is a bonafide record of research work carried out by him under my supervision.

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DECLARATION

I do hereby declare that this thesis entitled **“MARKOV DECISION PROCESS IN SUPPLY CHAIN MANAGEMENT”**has been originally carried out by me under the guidance and supervision of **Dr. C. ELANGO**, Associate Professor, PG and Research Department of Mathematical Sciences, Cardamom Planters’ Association College, Bodinayakanur – 625 513, Tamilnadu, India and that this work has not been submitted elsewhere for any other degree or diploma.

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P.K SANTHI

CONTENTS

Chapter 1

Introduction

This thesis deals with some stochastic inventory control problems in Supply Chain Management System (SCMS). The main objectives of the study are to find steady state probabilities of the inventory levels at various echelons of the system and the optimum values of the decision variables that minimize the total excepted cost in maintaining SCMS. Most of the results are illustrated with numerical examples and sensitivity analysis is done for various cost parameters such as holding cost, setup cost and backordering cost.

This introductory chapter contains a brief account of preliminary concepts in Inventory System and its developments, importance, classification, different types of decision making, inventory management, multi-commodity and Multi-echelon inventory system. The supply chain concept and its features in business are depicted in the literature relevant to the research work and an outline of the work done by the researcher has been given in the subsequent sections.

One of the seminar papers in the field of continuous review MEI is a basic one written by Sherbrook's in 1968. He assumed that (S-1,S) polices in a Deport of – Base system force reparable item used in American Air Force (AAF). He approximated the average unit years of inventory and stock art in the Air bases. This result has been used by many subsequent researchers, since it gives a nice approximations for the lead time of the bases. The lead time of bases has computational complexity in any MEI systems.

Moinzadeh, K., and Lee, H.L., (1986) [55] considered the issue of determining the optimal batch size for ordering and stock levels of the stocking locations by using a proper approximations. Moinzedah, K., (1987) [55] generalized previous models ME repairable inventory system to cover the cases of batch ordering and batch shipment. Deuermeyer, B., and Schwarz, L.B., (1981) [19] proposed simple approximations for a complex multi – echelon system (one – warehouse and multiple retailers) assuming backordering of all unsatisfied demands in all installations with batch ordering policy. Svoronos, A. and Zipkin, P., (1988) [90] proposed several refinements as the latter paper by second moment information (SD as well as mean) in their approximation. In 1990s Axsater, S., [8] provided a simple recursive procedure for determining the holding and stock out costs of a system consisting of one central warehouse and multiple retailers with (S-1,S) policy. Demands occurred during stock outs period are back ordered at all installations with constant lead times. Axsater, S., (1990) [8] proposed exact and approximate methods for evaluating previous systems for the case of general batch size in all installations but with identical retailers.

For the case of non – identical retailers, Axsater, S., (1993) [9] proposed methods for exact evaluations of two non – identical retailers and approximate evaluation for more than two non – identical retailers.

The common assumptions of the above papers are

(1) Demands during the stock out periods are backordered.

(2) Some of the conditions demands may be lost (partial back order).

Anderson (1991) [1] and Melchiors (2001) [54] have proposed an approximate method for the case of lost sales when the inventory control policy is (S-1, S) in all installations (one – warehouse and multiple retailers) and the unsatisfied demands are last in the retailer’s node.

They also introduced cost evaluations of such a system in the case of batch ordering policy as a future field of research.

Seifbarghy, M., and Jokar, M. R.A., (2005) [83] proposed a two echelon inventory system (one – warehouse and multiple retailers) with batch size determined by deterministic model with known replenishment costs at retailer and warehouse. They found optimal reorder levels, by minimizing the total holding costs of warehouse and retailers and stock out costs at the retailers.

1.1.1 Inventory System

Inventory Management is an integral part of Operations Research (OR) and is one of the vital area of applications of OR. Inventory theory, although originally suggested and motivated by practical problems has now been developed into a full-fledged Mathematical discipline. Mathematical models are available for different types of inventories such as raw materials, spare parts, cash and finished goods. Work in process or Inventory Control plays vital role in AI based manufacturing industries. Optimal decision problems are basic in inventory control systems.

The first quantitative analysis in inventory studies started with the work of Harris, F., [34] in 1915. He formulated mathematically a simple deterministic demand based an optimal inventory situation and obtained solution. This Economic Order Quantity (EOQ) was of practical value in 1950^s^. After the Second World War, several researchers like Pierre Massac (1946), Arrow, K.,, Harris, S., and Marschack, L., (1951) [7], Dvoretsky,A., Kiefer, J., and Wolfowitz, J., (1952) [20] and Whitin, T.M., (1953) [96] have discussed the stochastic nature of inventory problems.

A systematic analysis of (s, S) inventory model based on renewal theory was first provided by Arrow, K., Karlin, S., and Scarf, H., (1958) [6]. The book by Hadley and Whitin, T.M., (1963) [33], provides an excellent account of applications of inventory theory. A computational approach for finding optimal (s, S) inventory policies is given by Veinott, A.F., and Wagner, H.M., (1965) [94]. An excellent review by Veinott, A.F., (1966) [93], summarizes the status of mathematical theory of inventory with deterministic random demands until the early sixties. He focuses his attention on the determination of optimal policies of multi - item and / or for multi echelon inventory systems with certain and uncertain demands. The cost analysis of different inventory systems along with several other characteristics is given in Naddore, E., (1966) [56]. One of the oldest papers in the field of continuous review multi-echelon inventory system is a basic and seminal paper written by Sherbrooke, C., [85] in 1968. He assumed (S-1, S) policies in the Deport-Base systems for repairable items in the American Air Force and could approximate the average inventory and stock out level in bases.

Gross, D., and Harris, C.M., (1971) [31] developed continuous review (s, S) inventory models with state dependent lead times. Sivazlian, B.D., (1974) [86] considered a continuous review (s,S) inventory system with arbitrary inter arrival time distribution between demands, where each arrival demands exactly one unit. He obtained the transient and steady state distribution for the position inventory and shows that the limiting distribution of the position inventory is uniform and is independent of the inter arrival time distribution under many sharp assumptions. The same result for the case with arbitrarily distributed demand quantity has been obtained by Richard, F.R., (1975) [75]. An in-depth study of (s, S) inventory policy with arbitrarily distributed lead time is available in Srinivasan, S.K., (1979) [87].

The nature of inventory system consists of placing and receiving orders of given size Q = S – s with (s, S) policy for reorder. An inventory policy addresses two major issues, and elucidates answers to them. They are (i) How much to order? and (ii) When to order?. Answers to these questions need an elaborate cost analysis of the system.

Two major inventory systems classified according to the method of observing the inventory process are **periodic** and **continuous review** systems. The former system has the characteristics of multi-stage decision process at discrete time points, normally equally spaced. If the times between the consecutive demands are random, then the review time points are not equally spaced.

In continuous review system, every demand occurred is noticed and transaction is reported to update the inventory level. So, it is also termed as transaction reporting system. Throughout this thesis, we call this system simply continuous review inventory system. Because of the advent of information technology and the use of electronic data processing machines in inventory management, the continuous review system has received more attention.

The periodic review inventory system is considered as a multi-stage decision making system, for which dynamic programming technique provides suitable tools to obtain optimal decision rule. However, for continuous review systems, this technique is not so much adaptive, since it yields complicated expressions.

1.1.2 Costs Associated with Inventories

Various costs associated with inventory control system are often classified as follows:

Set–up cost

This is the cost associated with the setting up of machinery before starting production. Set–up cost is generally assumed to be independent of the quantity ordered for or produced.

Ordering cost

Ordering cost is a cost associated with ordering of raw material for production purposes. Advertisements, consumption of stationary and postage, telephone charges, telegrams, rent for space used by the purchasing department, travelling expenditures incurred, etc., constitute the ordering cost.

Purchase (or production) cost

The cost of purchasing (or producing) an unit item is known as purchase (or production) cost. The purchase price will become important when quantity discounts are allowed for purchases above a certain quantity or when economics of scale suggest that the per unit production cost can be reduced by a larger production run.

Carrying (or holding) cost

The carrying cost or holding cost is associated with carrying (or holding) inventory. This cost generally includes the costs such as rent for space use of storage, interest on the money locked-up, insurance of store equipment, production, taxes, depreciation of equipment and furniture used, etc.

Shortage (or stock- out) cost

The penalty cost for running out of stock (i.e., when an item cannot be supplied on the customer's demand) is known as shortage cost. This cost includes the loss of potential profit through sale of items and loss of goodwill in terms of permanent loss of customers and its associated lost profit in future sales.

Salvage cost (or selling price)

When the demand for certain commodity is affected by the quantity stocked, decision problem is based on a profit maximization criterion that includes the revenue from selling.

Salvage value may be combined with the cost of storage and hence is generally neglected. The revenue lost after selling the items below the cost of holding the items is salvage cost.

Revenue cost

When it is assumed that both price and demand of the product are not under control of the organization, the revenue from the sales is independent of the company's inventory policy and may be neglected except for the situation when the organization cannot meet the demand and the sale is lost. Therefore, the revenue cost may or may not be included in the study of inventory policy.

Re-order level

The level between maximum and minimum stock at which purchasing (or manufacturing) activities must start for replenishment is known as re-order level. In the care of (s, S) policy, s is the reorder level, with ordering quantity $Q = S - s$.

Replenishment

Although an inventory problem may operate with lead time, in some cases replacement of stock may occur instantaneously or uniformly. Replenishment with lead time occurs in case the stock is purchased from outside source whereas uniform or instantaneous replenishment may occur when the product is manufactured by the company.

Queuing System:

Queuing System is a phenomena where customers arrive for service. They may wait in the queue for getting service if it is not immediately available. Customers will leave the system after being served. The term customer may refer to a *person, car, a message or an aero plane*. So customer in the queue may be classified into different categories:

(i) having independent decision making power (ii) controlled decision by the external entity (agent) or (iii) decision on customer is taken by the system manager.

The above category of customer's behavior leads to the events (i) reneging (ii) balking and (iii) jockeying which is optional / compulsory.

The general queuing system (single server) can be described as in fig.[i]. Other variants of queuing models are (i) multiple server (ii) independent class of customers are also available.

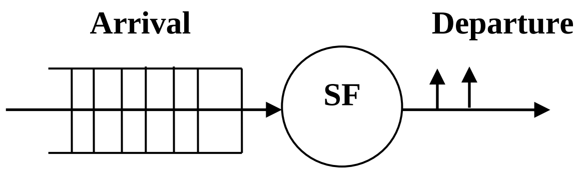


fig.[i]

Queuing theory was developed to provide Mathematical models to study the behaviour of systems that attempts to provide service for randomly arriving customers. The pioneer investigator of these systems was the Danish engineer A.K.Erlang (1909) [25]. He published his seminal paper "The theory of probabilities and telephone conversations". He observed that most common arrival to telephone station is Poisson and the service time is almost content. Other researchers in queuing systems are Thompton Fly, Felix Pollaczek, Kolmogorov, Khintchine, Cromm and Palm.

There are many notable applications of theory of queues till date but most of them have little prediction value. All these works are documented in the literature of probability, operations research, management science and industrial engineering. Some examples are traffic flow, scheduling, supply chain management and facility design.

The following six characteristics of queuing systems provide the systematic description of queue.

(i)arrival pattern of customers, (ii)service pattern of servers, (iii) number of servers, (iv) system capacity, (v) number of service stages. (vi) queue discipline.

Notations:

Queuing systems are well categorized and notations are fixed by Kendall (1953) [48], which is now rather standard throughout the queuing literature. The queuing process is described by a symbols and slashes such that (A/B/X/Y/Z). For example (M/M/I/∞/FCFS) is a single server queuing system with Poisson arrival and exponential (memory less) server time, and FCFS discipline with infinite system capacity.

System performance measures give the effectiveness of queuing system. In general, there are three types of system performance measures namely, (i) system size (ii) waiting time of a typical customer and (iii) the ideal time of the servers. Since most of the queuing systems have stochastic elements such as arrival process, service process and customer behavior. These measures are often random variables and their expected values are desired.

The task of the queuing analyst is generally of two types.

(i) to determine the value of appropriate measures of effectiveness for a given process.

(ii) to design an optimal queuing system subject to a specific criterion.

In any case, the analyst will strive to solve this problem by analytical means. If he fails to get analytical solution, the alternative method is simulation. Ultimately, the issue comes down to the problem of trade-off of better customer service versus providing more service capacity.

Some general results on Queuing:

According to Kendall's [48] notation, G/G/1 and G/G/C are most general queuing system of single and multiple servers where G denotes the general distributation with mean m. Suppose the average rate of customers entering the queuing system is λ and the average rate of serving customers is μ. The measure of traffic congestion for a c-server system is $\rho = \frac{\lambda}{c\mu}$ (traffic intensity).

If $\rho > 1$, that is, $\lambda > c\mu$ there is a huge congestion in the system as time goes on. If $\rho < 1$, i.e., $\lambda < c\mu$, the system will be in control and steady state exist in the long run. In the case if $\rho = 1$, i.e., $\lambda = c\mu$, will lead to perfect balance exists in the deterministic values of λ and μ, but in the case of stochastic elements of λ and μ, till a grow up in the queue length is expected.

Suppose N(t) denote the total number of customers in the system (queue + server) $N(t) = N_q(t) + N_s(t)$, then define the following probability expressions.

$p_n(t) = \Pr\{N(t) = n\}$ and $p_n = \Pr\{N = n\}$ is the steady state.

In the c-server case the two expected namely numbers of customers in systems and in the queue are given by

$$L = E[N] = \sum_{n=0}^{\infty} n p_n \quad \text{and} \quad L_q = E[N_q] = \sum_{n=c+1}^{\infty} (n-c) p_n$$

Another general result derived by D.C.Little (1960) [50] depicts a direct relationship between L, L_q, W and W_q as follows.

$L = \lambda W$ and $L_q = \lambda W_q$ where $W = E[T]$ and $W_q = E[T_q]$, where T_q represent the time the customer spends waiting in the queue prior to entering service and T represent the total time a customer spends in the system: ($T = T_q + S$, where S is the service time) for a single customers) .Here T, T_q and S are random variables.

1.2 Supply Chain

A supply chain is a network of facilities and distribution options that performs the functions of procurement of materials, transformation of these materials into intermediate and finished products, and the distribution of these finished products to customers. Supply chains exist in both service and manufacturing organizations, although the complexity of the chain may vary greatly from industry to industry and firm to firm.

Supply chain integration is concerned with functional integration and co-ordination among the supply chain partners providing optimal cost. In an independently managed supply chain, each member in each stage will optimize his own operational costs in a decentralized fashion. Generally, it has been realized that such managerial independence of the supply chain partners may increase the imbalance between demand and supply. Such independence also has been recognized as direct cause of increased costs. This pushed firms towards the full integration of the supply chain resources and the proper co-ordination of decisions. Researchers in SCM report that closer collaboration among the chain partners, increased level of information sharing, and high level of co-ordination of various decision processes lead to improved customers service and reduced costs. The significant advances in information and communication technologies facilitated the provision and sharing of the business information necessary for efficiency improvement. This, in turn, facilitated the development in the integrated supply chain management.

1.2.1 Supply Chain Management

Supply Chain Management (SCM) is an essential element to operational efficiency. SCM can be applied to customer satisfaction and company success, as well as within societal settings, including medical missions; disaster relief operations and other kinds of emergencies; cultural evolution; and it can help improve quality of life. Because of the vital role SCM plays within organizations, employers seek employees with an abundance of SCM skills and knowledge. Supply chain management is critical to business operations and success for the following reasons:

Efficient supply chain management ensures prompt delivery of goods and services. The importance of Supply Chain Management thus is in:

- Reduced inventories along the chain

- Better information sharing among the partners
- Planning being done in consultation rather than in isolation

The benefits too would be reflected in terms of (1) Lower cost (2) Better customer service, (3) Efficient manufacturing (4) Better trust among the partners leading to win-win situation.

1.2.2 Classification of supply chain management

(a) Classification based on structure:

There are different structures of multi-echelon inventories. The simplest system is a ***Series system*** where two or more inventories are coupled. This system can be encountered in different situations. The other types are ***Distribution system*****,** ***Assembly system*** and ***General system*****.**

(i) Serial two-echelon system

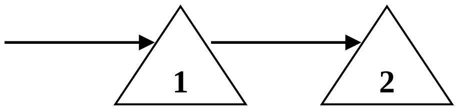


Figure 1.1 - A series two-echelon inventory system

For instance where there are only two echelons and only a single installation at each echelon the first inventory holding the stock is used to periodically replenish the inventory at the second. Then is also called tandem supply chain (two echelon)

(ii) Serial multi-echelon system:

A serial system with more than two echelons is known as multi-echelon system. An illustration is given in figure 1.2, where installation 1 has its inventory replenished periodically, then the inventory at installation 1 is used to replenish the inventory at installation 2 periodically, then installation 2 does the same for installation 3 , and so on down to the final installation(say N). This structure is called tandem SCM with multiple echelon.

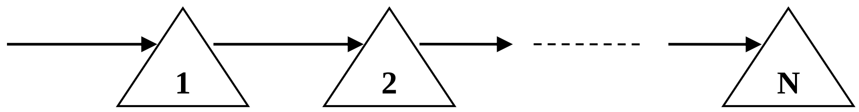


Figure 1.2 - A series multi-echelon inventory system

(iii) Distribution system

The distribution system in SCM is divergent system in which each stocking point has at least one predecessor. A typical situation in practice is that one central warehouse supplies goods to several retailers following through the distribution centre.

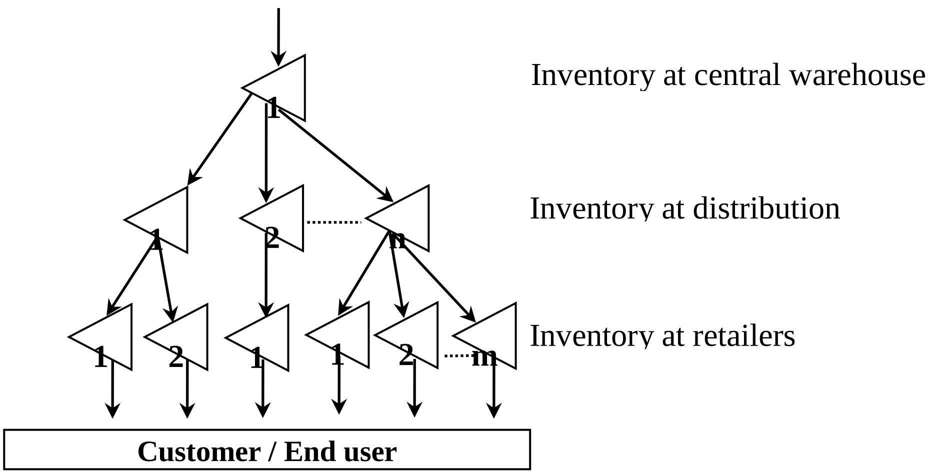


Figure 1.3 three-echelon distribution inventory system

(iv) Assembly system

Assembly system in convergent system which is just the opposite of a general distribution system. Each stocking point has atleast one immediate successor, in which when a sub- assembly plant receives its components from the multiple suppliers or when a factory receives its sub-assemblies from multiple subassembly plants. Such structure holding inventory in SCM is called an assembly system.

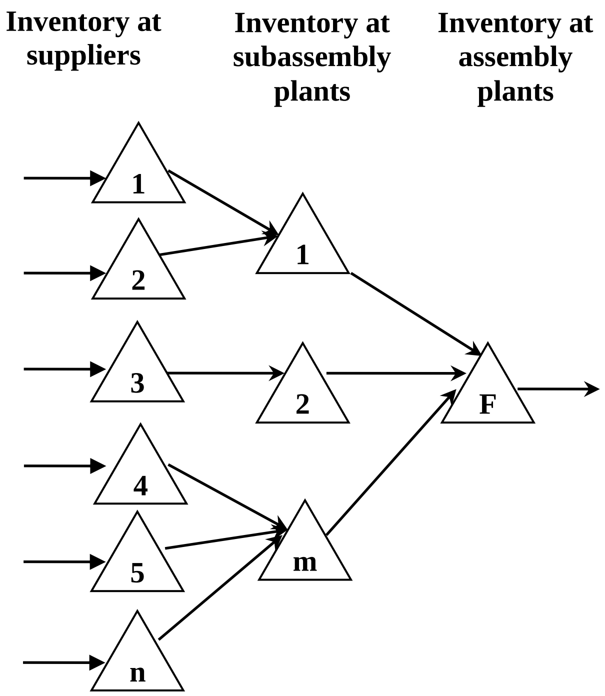


Figure 1.4 a three-echelon assembly inventory system

(v) General systems:

General systems in a supply chain can of course be of more complex nature and a combination of different systems described above. The figure 1.5 illustrates a general system which is a complex structure. More complex inventory systems are called multi-echelon inventory systems. An echelon in this case is referred to a single level in a supply chain (collection of entities in the same level). In multi-echelon inventory systems several stocking points are evaluated together comparing to single echelon control where each level is evaluated independently.

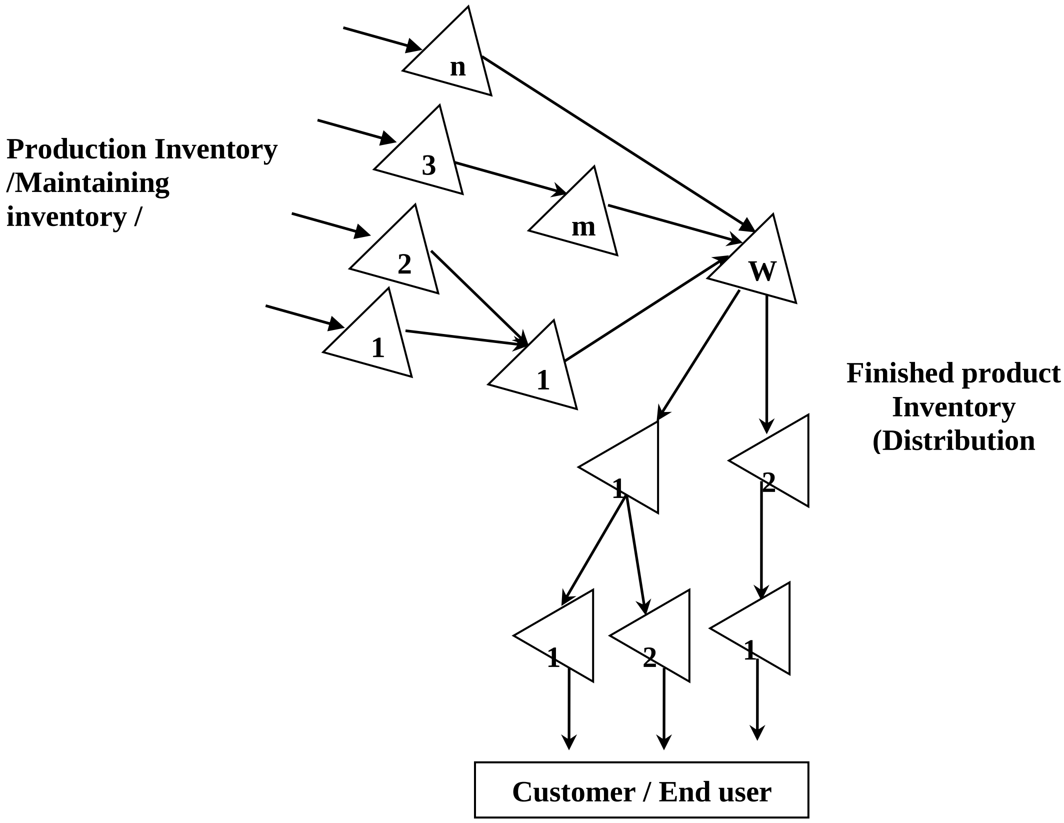


Figure 1.5 general inventory system

(b) Classification based on decisions:

The decisions for a supply chain system are classified into two categories: ***strategic*** and ***operational***. Strategic decisions are made typically over a longer time horizon. The function of operational decisions exists short-term, and focus on activities over a day-to-day basis. The effort in these types of decisions is to effectively and efficiently manage the product flow in a "strategically" planned supply chain.

There are four major decision areas in supply chain management and they are listed below:

- Location decision
- Production decision
- Inventory decision
- Transportation decision

Inventory decision is an important component of the supply chain management, because inventories exist at each and every stage of the supply chain as raw material or semi-finished or finished goods. They can also be as Work-in-process between the stages or stations. Since holding of inventories can cost anywhere between 20% to 40% of their value, their efficient management is critical in Supply Chain (SC) operations. In this thesis we studied a special trail of inventory management in tandem supply chain.

The integrating of inventory decisions in supply chain systems is an important issue. The system, which prevents an excess inventory, reflects the lack of planning, poor communication and management needs. Thus ***information sharing*** is also important in supply chain management system

The power of *information sharing driver* grows stronger in each year as the technology for collecting and sharing information becomes more widespread, easier to use, and less expensive. Information, much like money, is a very useful commodity because it can be applied directly to enhance the performance of the other four supply chain drivers. High levels of responsiveness can be achieved when companies collect and share accurate and timely data generated by the operations of the other four drivers. The supply chains that serve the electronics markets are some of the most responsive in the world. Companies in these supply chains from manufacturers, to distributors, to the big retail stores collect and share data about customer demand, production schedules, and inventory levels.

1.2.3 Inventory Control in Supply Chain

Inventory exists in the supply chain because of a mismatch between supply and demand. This mismatch is intentional at a steel manufacturer where it is economical to manufacture in large lots that are then stored for future sales. The mismatch is also intentional at a retail store where inventory is held in anticipation of future demand or to attract more customers. An important role that inventory plays in the supply chain is to increase the amount of demand that can be satisfied by having product ready and available when the customer wants it. Another significant role inventory plays is to reduce cost by exploiting any economics of scale that may exist during both production and distribution.

Inventory is spread throughout the supply chain from raw materials to work in process from finished goods, to suppliers, manufacturers, distributors, and retailers hold. Inventory is a major source of cost in a supply chain and it has a huge impact on responsiveness and cost of operations of SC. If we think of the responsiveness spectrum, the location and quantity of inventory can move the supply chain from one end of the spectrum to the other. For example, an apparel supply chain with high inventory levels at the retail stage has a high level of responsiveness because a customer can walk into a store and walk out with the shirt they were looking for. In contrast, an apparel supply chain with little inventory would be very unresponsive. A customer wanting a shirt would have to order it and wait several weeks or even months for it to be manufactured, depending on the basis of little inventory existed in the supply chain.

Inventory also has a significant impact on the material flow time in a supply chain. Material flow time is the time that elapses between the points at which material enters the supply chain to the point at which it exist. Another important area where inventory has a significant impact is throughout.

1.2.4 Inventory Management in SCM

Every company has the challenge of matching its supply volume to customer demand. How well the company manages this challenge has a major impact on its profitability. Open Standards data shows that the median company carries an inventory of 10.6 percent of annual revenues. The typical cost of carrying inventory is at least 10.0 percent of the inventory value. So the median company spends over 1 percent of revenues carrying inventory, although for some companies the number is much higher.

Also, the amount of inventory held has a major impact on available cash. With working capital at a premium, it's important for companies to keep inventory levels as low possible and to sell inventory as quickly as possible. When Wall Street analysts look at a company's performance to make earnings forecasts and buy and sell recommendations, inventory is always one of the top factors they consider. Studies have shown a 77 per cent correlation between overall manufacturing profitability and inventory turns.

(a) Inventory optimization - Deterministic / Stochastic

Inventory optimization models can be either **deterministic** - with every set of variable states uniquely determined by the parameters in the model or **stochastic** - with variable states described by probability distributions. Stochastic optimization also accounts for demand volatility which is priority No. 1 among the challenges faced by supply chain professionals.

1.2.5 Inventory optimization benefits in SCM

Several companies have achieved financial benefits by employing inventory optimization. A study by IDC Manufacturing Insights found that many organizations that utilized inventory optimization reduced inventory levels by up to 25 per cent in one year and enjoyed a discounted cash flow above 50 per cent in less than two years. Electro components, a United Kingdom based world's largest distributor of electronics and maintenance products, increased profits by £36 million by using inventory optimization to achieve higher service levels while reducing inventory. BP-Castrol used inventory optimization to reduce finished goods inventory by an average of 35 per cent in two years while increasing service levels (defined as line fill rates) by 9 per cent. Smiths Medical, a division of Smiths Group, used inventory optimization to better address demand volatility and supply variability, thus reducing the risk of both under-stocks and overstocks while smoothing out manufacturing cycles.

1.2.6 Inventory control policy in SCM

Two fundamental questions that must be answered in controlling the inventory of any item are when and how much should be ordered for replenishment. The answers of these questions determine the inventory control policy. In the face of uncertainty it is more complicated to determine the inventory control policy. The most frequent uncertain factor is the stochastic nature of customer demand.

A continuous-review policy requires knowledge of the inventory level at all times. If the customer demand is one unit, the ordering size (Q) is always equal to S-s. Therefore, the (s, S) policy changes into the well known continuous review (r, Q) policy, where r=s is the re-order point. If the ordering cost is negligible in the (r, Q) policy, then the policy is called one-for-one, (S-1, S), or base stock policy (Hadley and Whitin 1963 [33]).

1.2.7 Perishable Inventory System in SCM

Perishable goods are those goods, which have a fixed or specified lifetime after which they are considered unusable, i.e., they cannot be utilized to meet the demand. The planning and control of perishable inventory systems is important because in real life products like milk, blood, drug, food, vegetables and some chemicals do have fixed life times after which they will perish. The presence of these kinds of products after their lifetime will not only occupy space of the store but also effect the lifetime (damage) of the neighboring items. In some cases of perishable goods, which consume electricity for their storage, the loss is greater. The determination of the parameters of inventory models, to meet the demand of these types of goods hence becomes very crucial. The problem becomes difficult when there are stochastic demands and lead times. In this thesis we assume uncertain life time goods which are maintained in a supply chain.

1.2.8 Multi-echelon inventory management in SCM

The technical term for supply chain inventory management is multi-echelon distribution inventory system. For over 50 years, researchers have been addressing a variety of problems in multi-echelon inventory systems which has been broadly categorized as serial or assembly systems.

The multi-echelon inventory management problem has been extensively studied in the past several decades. However, most of the inventory literature only considers the optimization of inventory decisions without integrating them with supply chain design and planning decisions. Simpson first studied the control of multi- echelon inventory system for a serial supply chain. In that paper, Simpson proposes the guaranteed service approach to describe the mechanics of the serial inventory system, in which each stage operates a base-stock policy in the face of random but bounded demand. Simpson's results showed that the optimal inventory policy for the serial system is an "all or nothing" policy, i.e. each stage either has no safety stock, or carries enough stocks to decouple the downstream stages form the upstream stages. Almost at the same time, Clark and Scarf [18] also studied the optimal control policy of the multi-echelon inventory system with each stage operating with a base-stock policy. In contrast to Simpson's work, Clark and Scarf [18] consider that the uncertain demand at each stage to be unbounded, and thus the service level and replenishment lead time of each stage depend on the upstream adjacent supply stages' stock level. To solve the problem, they proposed a dynamic programming approach based on the calculation of recursions for stage inventory and replenishment amount. Later, Federgruen and Zipkin,P.

(1984) [26] extended the result of Clark and Scarf (1960) [18] to infinite horizon, and showed that a stationary order-up-to level echelon policy is optimal.

1.3 Packages used

(a) MATLAB

The name MATLAB stands for MATrixLABoratory. MATLAB is a high-performance language for technical computing. It integrates computation, visualization, and programming environment. Furthermore, MATLAB is a modern programming language environment: it has sophisticated data structures, contains built-in editing and debugging tools, and supports object-oriented programming. It also has easy to use graphics commands that make the visualization of results immediately available. These factors make MATLAB an excellent tool scientific research.

MATLAB is widely used in all areas of applied mathematics, in education and research at universities, and in the industry. MATLAB software is built up around vectors and matrices. This makes the software particularly useful for linear algebra but MATLAB is also a great tool for solving algebraic and differential equations (symbolic) and for numerical integration (numeric). MATLAB has powerful graphic tools and can produce nice pictures in both 2D and 3D. It is also a programming language, and is one of the easiest programming languages for writing Mathematical programs. MATLAB also has some tool boxes useful for signal processing, image processing, optimization, etc.

(b) MAPLE

Maple is a powerful mathematical software package. It can be used to obtain both symbolic and numerical solutions of problems in arithmetic, algebra, and calculus and to generate plots of the solutions it generates.

1. 4 Outline of the present work

This thesis is divided into seven chapters including this introductory chapter. (**Chapter-1)** This chapter presents some preliminary concepts in Inventory System and its developments, importance and different type of decision-making. The details of software used in the computational part of the thesis are also given.

In **Chapter-2,** the tools and techniques used to solve Markov Decision Problems are listed. Major and versatile tools used are Markov Chains (continuous and discrete time) and relevant results. (Definition, Examples and Theorems). Markov Decision Process and its solution procedures like (1) LPP (2) Policy iteration and (3) Value iteration are described in detail.

Chapter-3, deals with continuous time Markov decision processes with finite state and finite action set. The optimization criteria is uses expected average cost rate. LPP is used to solve the decision problem and optimal policy is obtained to run the supply chain management system.

In **Chapter-4**, semi Markov decision problem on SCM is studied in that which has distribution centre and RV with service facility. This chapter differs from previous one by maintaining perishable inventory in both DC and RV nodes. Inter arrival time of demands as RV is exponentially distributed with parameter λ. The service time is exponentially distributer with parameter δ. Maximum capacity of waiting space is η. Each item in RV has the perishable rate Q with exponentially distribution.

Chapter-5, deals with as a continuous time Markov decision on process with state space and finite action space. The service rate at retailer node is controlled by having decision according to the long run expected cost rate criteria. Each item in inventory has the perishing rate Q with exponentially distributed life time. Impatient customers range from the system with rate $\alpha > 0$. (Exponentially distributed inter range time).

In **Chapter-6,** we consider a different model in which inventory is controlled at retailer node but no service facility (items are issued as and when demand occurred). That is there is no queue in the retailer node. Demands that are occurring during the stock out periods are backlogged upto as specified queuing 'b' are retailer node such that the ordering quantity Q > b + s**.**

Chapter-7, discusses we consider a different kind of supply chain which maintains perishable inventory at the retailer node. The demand process is assumed to be Poisson with mean rate λ. Inventory at distribution centre is maintained in terms of maximum S items is stored, where Q = S – s items are ordered at reorder point s.. The ordering policy at retailer node is (s, S) type and ordering quantity Q = S – s > b + s, where b is a batch order limit. Back logging is assumed up to a finite level say 'b' (>0)

The thesis ending with references, list of research papers presented / publication research papers.

CHAPTER-2

Analytical Tools and Techniques

2.1 Introduction:

In probability theory, a stochastic process or sometimes random process (widely used) is a collection of random variables; this is often used to represent the evolution of some random value, or system, over time. This is the probabilistic counterpart to a deterministic process (or deterministic system). A stochastic process is one whose behavior is non-deterministic; it can be thought of as a sequence of random variables. Any system or process that can be analyzed using probability theory is stochastic. Stochastic system and processes play a fundamental role in mathematical modules of phenomena in many fields of science, engineering, and economics.

2.2 Stochastic processes

A random process (or) a *stochastic process* is a collection of random variables $\{ X_n, n \geq 1)$ on a same sample space. Some times n starts from 0. That is $\{ X_n, n \geq 0\}$, with initial random variable X_0.

The set of all possible values of a single random variable X_n in the set $\{X_n, n\geq 1\}$ is known as its state space.

If the state space of a stochastic process is discrete, it is called a discrete state space process, or a chain. If the state space is continuous, then the stochastic process is called as a continuous state process.

We can classify the stochastic process into discrete parameter process or continuous parameter process according to the parameter T which is discrete or continuous.

2.2.1 Types of stochastic processes

One dimensional process can be classified into the following types of processes:

1. Discrete time, Discrete state space *(DTDS)*

2. Discrete time, Continuous state space *(DTCS)*

3. Continuous time, Discrete state space *(CTDS)*

4. Continuous time, Continuous state space *(CTCS)*

2.2.2 Definitions

(a) Independent stochastic process

If the finite dimensional joint distribution of a stochastic process $\{X(t)|t \in T\}$ satisfies : $F_n(x_1,x_2,\ldots\ldots x_n) = \Pr\{X(t_1)\leq x_1, X(t_2)\leq x_2,\ldots\ldots\ldots X(t_n)\leq x_n\}$

$$= \prod_{i=1}^{n} \Pr\{X(t_i) \leq x_i\}$$ for $0\leq t_1\leq t_2\leq \ldots\leq t_n$, then the stochastic process$\{X(t); t\in T\}$ is called an independent stochastic process.

(b)Reachable State

Let $\{Xt: t \in T\}$ is a stochastic process with state space E.A state $\in T$ j is *reachable* from state i if there is a path starting from i to j.

(c) Transient State

A state $\in E$ is a *transient state* if there exists a state j that is *reachable* from i, but the state i is not reachable from state j .

(d) Markov Chain

The stochastic process $\{X_n, n \geq 0\}$ is called a *Markov chain*, if for $j,k,j_0,\ldots.j_{n-2} \in E$ (or any subset of E)

- for all $n \geq 0$, $Xn \in E$ with probability 1,
- $Pr\{X_n=k \mid X_{n-1}=j, X_{n-2}=j_1,\ldots.X_0=j_{n-1}\} = Pr\{X_n=k \mid X_{n-1}=j\} = p_{jk}$ (say)

whenever the first member is defined.

(e) Conditional probability for a Markov Chain

The conditional probability for a Markov Chain $Pr\{X_n=k \mid X_{n-1}=j\} = p_{jk}$, $j,k \in N$, is independent of n. The probability P_{jk} are called the transition probabilities for the *Markov chain.*

2.3 Continuous Time Markov Chains

A continuous time stochastic process $\{X(t'); t' \geq 0\}$ has the *Markovian property* if $P\{X(t + s) = j/X(s) = i \text{ and } X_0 = x_0\} = P\{X(t+s) = j/X(s) = i\}$, for all $i, j = 0,1,\ldots..,M$ and for all $t > 0$ and $s \geq 0$.

2.3.1 Definitions

(a) Stationary transition probabilities

If the transition probabilities are independent of s, so that

$P_r\{X(t+s) = j/X(s) = i\} = P\{X(t) = j/X(0)=i\}$ for all $s>0$, they are called *stationary transition probabilities*

(b) Transition probability matrix

The transition probabilities p_{jk} satisfy the relations $p_{jk} \geq 0$, $\sum p_{jk} = 1$ for all j.

These probabilities may be written in the matrix form, $\{X_t : t \in T\}$ and $E = \{1,2,3,\ldots\}$

$$P=\begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & \cdots \\ p_{21} & p_{22} & p_{23} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

This is called the transition probability matrix (or) matrix of probabilities (t.p.m) of the Markov Chain. P is a stochastic matrix. i.e a square matrix with non-negative elements and unit row sums.

(c) Accessible

A state j is said to be accessible from state i if for some $n \geq 0, p_{ij}^{(n)} > 0$ and can be denoted as $i \rightarrow j$.

(d) Communication

A state i is said to communicate with state j if there exists an integer m and n such that $p_{ij}^{(n)} > 0$ and $p_{ij}^{(m)} > 0$. That is $i \rightarrow j$ and $j \rightarrow i$ occurs.

In general,

1. Any state communicates with itself (because $p_{ii}^{(n)} = 1 > 0$).
2. If state i communicates with state j, then state j communicates with state i. that is $i \leftrightarrow j$
3. If state i communicate with state j and state j communicates with state k, then state i communicates with state k.(i.e. $i \leftrightarrow j$, $j \leftrightarrow k$ then $i \leftrightarrow k$)

(e) Aperiodic

State $i \in E$ is said to be recurrent if $p_{ii}^{(m)} > 0$ for some $m \geq 1$. The period d_i, the time of a return to state i is defined as the greatest common divisor of all m such that $p_{ii}^{(m)} > 0$.

Thus, $d_i = G.C.D\{m : p_{ii}^{(m)} > 0\}$; State i is said to be aperiodic if $d_i = 1$ and periodic if $d_i > 1$. Clearly state i is aperiodic if $p_{ii} \neq 0$.

(f) Communicating class

A set $C \subset S$ is said to be a communicating class if $i \leftrightarrow j, \forall\ i, j \in C$

(g) Closed

A communicating class C is said to be closed if $i \leftrightarrow j, \forall\ i, j \in C$ and $k \notin C$ then k cannot be accessed from i.

(h) Irreducible

If the Markov Chain (MC) does not contain any other proper closed subset other than the state space, then the MC is called irreducible, and the corresponding chain is called unichain.

2.4 First passage time distribution

Suppose that a system starts with the state j. let $f_{jk}^{(n)}$ be the probability that it reaches the state k for the first time at the nth step (or after n transitions) and let $p_{jk}^{(n)}$ be the probability that it reaches state k (not necessarily for the first time) after n transitions.

Let F_{jj} denote the probability that start with state j, the system will ever reach state j. Clearly $F_{jj} = \sum f_{jj}^{(n)}$.

Let F_{jk} denote the probability that start with state j, the system will ever reach state k. clearly

$$F_{jk} = \ {}_{n=1}^{\infty} f_{jk}^{(n)}$$

We have $\sup_{n\geq 1} p_{jk}^{(n)} \leq F_{jk} \leq \ {}_{m\geq 1} p_{jk}^{(m)}$, for all $n \geq 1$.

We have to consider two cases, $F_{jk} = 1$ and $F_{jk} < 1$.

When $F_{jk} = 1$, it is certain that the system starting with state j will reach state k; in this case $f_{jk}^{(n)}, n = 1,2, \ldots.$ is a proper probability distribution and this gives the first passage time distribution for k, given that the system starts with j.

The mean (first passage) time from state j to state k is given by

$$\mu_{jk} = \;_{n=1}^{\infty} n\, f_{jk}^{(n)}$$

In particular, when k = j, $f_{jk}^{(n)}, n = 1,2, \ldots.$ will represent the distribution of the recurrence times of j; and $F_{jj} = 1$ will imply that the return to the state j is certain.

In this case, $\mu_{jj} = \;_{n=1}^{\infty} n\, f_{jj}^{(n)}$ is known as the mean recurrence time for state j.

Thus, two questions arise concerning state j: first, whether the return to state j is certain and secondly, when this happens, whether the mean recurrence time μ_{jj} is finite or not.

2.4.1 Definitions

Consider a Markov Chain $\{X_n, n \geq 0\}$ with state space E.

(a) Mean recurrence

Let $j \in E$, then the mean recurrence time of state j is given by

$\mu_{jj} = \sum n f_{jj}^{(n)}$ when $f_{jj}^{(n)}$ is known as the probability distribution.

(b) Persistent

A state $j \in E$ is said to be persistent (recurrent) if $F_{jj} = 1$(i.e. return to state j is certain) and transient if $F_{jj} < 1$.

A persistent state j is said to be null persistent if $\mu_{jj} = \infty$, i.e. if the mean recurrence time is infinite and is said to be non-null or positive persistent if $\mu_{jj} < \infty$.

A persistent non-null and aperiodic state of a Markov Chain is said to be Ergodic. If all states of a MC are ergodic, then the Markov Chain is called ergodic chain.

(c) Absorbing

A state $i \in E$ is an absorbing state if and only if $p_{ii} = 1$.

2.5 Theorems

2.5.1 First Entrance Theorem

Let $\{X_n, n \geq 0\}$ be a Markov Chain with state space E. Then for any states j and k. $p_{jk}^{(n)} = \sum_{r=0}^{n} f_{jk}^{(r)} p_{kk}^{(n-r)}$, $n \geq 1$, $0 \leq r \leq n$

with $p_{kk}^{(0)} = 1, f_{jk}^{(0)} = 0, f_{jk}^{(1)} = p_{jk}$.

Proof:

Intuitively, the probability that starting with j, state k is reached for the first time at the r^{th} step and again after that at $(n-r)^{th}$ step is given by $f_{jk}^{(r)} p_{kk}^{(n-r)}$ for all $r \leq n$. These cases are mutually exclusive events. The total probability that transition of j to k in n step is given by

$p_{jk}^{(n)} = \sum_{r=0}^{n} f_{jk}^{(r)} p_{kk}^{(n-r)}, n \geq 1$. This result can also be written as $p_{jk}^{(n)} = \sum_{r=0}^{n-1} f_{jk}^{(r)} p_{kk}^{(n-r)} + f_{jk}^{(n)}, n > 1$

2.5.2 Theorem :

Let $\{X_n, n \geq 0\}$ be a Markov Chain with state space E. Then state j is persistent if and only if $\sum p_{jj}^{(n)} = \infty$.

Proof: Let $P_{jj}(s) = \sum_{n=0}^{\infty} P_{jj}(s) s^n = 1 + \sum_{n=0}^{\infty} P_{jj}(s) s^n$, $|s| < 1$

$F_{jj}(s) = \sum_{n=0}^{\infty} f_{jj}^{(n)} s^n = \sum_{n=1}^{\infty} f_{jj}^{(n)} s^n$, $|s| < 1$

be the generating functions of the sequences $p_{jj}^{(n)}$ and $f_{jj}^{(n)}$ respectively.

We have $p_{jk}^{(n)} = \sum_{r=0}^{n} f_{jk}^{(r)} p_{kk}^{(n-r)}, n \geq 1$

Multiplying both sides of the above equation by s^n and adding for all n≥1, we get

$$P_{jj}(s)\text{-}1 = F_{jj}(s) P_{jj}(s)$$

The right hand side of the above is immediately obtained by considering convolution of sequence f_{jj} and p_{jj} and that the generating function of the convolution is the product of the two generating functions.

Thus we have

$$P_{jj}(s) = \frac{1}{1-F_{jj}^{(s)}} \ , \quad s < 1$$

Assume that state j is persistent which implies that F_{jj} =1.

Using Abel's lemma, we get

$$\lim_{s\to 1} F_{jj}(s) = 1$$

Thus

$$\lim_{s\to 1} p_{jj}(s) \to \infty$$

Since the coefficients of $p_{jj}^{(s)}$ are non-negative Abel's lemma applies and we get

$\sum_{n=0}^{\infty} p_{jj}^{(n)} = \infty$

Conversely, assume that $\sum_{n=0}^{\infty} p_{jj}^{(n)} = \infty$ and suppose state j is transient then by Abel's lemma, we get $\lim_{s\to 1} F_{jj} \quad s < 1$ and $\lim_{s\to 1} p_{jj}(s) < \infty$

Since the coefficients $p_{jj}^{n} \geq 0$, we get $\sum_{n=1}^{\infty} p_{jj}^{(n)} < \infty$, which is contradiction. Hence the result.

Remarks:

- State j is transient if $\sum p_{jj}^{(n)} < \infty$; this implies that if j is transient then $p_{jj}^{(n)} \to 0$ as $n \to \infty$.
- The state of a finite Markov Chain must contain at least one persistent state
- If k is a transient state and j is an arbitrary state then $\sum p_{jk}^{(n)}$ converges and $\lim p_{jk}^{(n)} \to 0$.
- If a Markov chain having a set of transient states T, starts in a transient state, then with probability 1, it stays at the transient set of states T for only a finite number of times after which it enters a recurrent state where it remains forever.

2.5.3 Theorem:

Let $\{X_n, n \geq 0\}$ be a Markov Chain with state space E, if state $j \in E$ is persistent non – null, then as $n \to \infty$.

1. $p_{jj}^{(nt)} \to t/\mu_{jj}$, when state j is periodic with period t;

and

2. $p_{jj}^{(n)} \to 1/\mu_{jj}$, when state j is aperiodic.

In case state j is persistent null, (whether periodic or aperiodic), then

$p_{jj}^{(n)} \to 0$, as $n \to \infty$.

Proof :

Let state j be persistent; then define $\mu_{jj} = \sum_n n f_{jj}^{(n)}$,

We may put f_{jj}^{n} for f_n, p_{jj}^{n} for u_n ,and μ_{jj} for μ.

We have

$$i \quad p_{jj}^{nt} \rightarrow t \quad \mu_{jj} \text{, as } n \rightarrow \infty \quad \text{when state j is periodic with period t.}$$

When state j is aperiodic (i.e. t=1), then

$$ii \quad p_{jj}^{nt} \rightarrow 1 \quad \mu_{jj} \text{, as } n \rightarrow \infty$$

In case state j is persistent null, $\mu_{jj}=\infty$, and $p_{jj}^{(n)} \rightarrow 0$ as $n \rightarrow \infty$

Note:

1) If j is persistent non – null, then lim $p_{jj}^{(n)} > 0$
2) If j is persistent null or transient then lim $p_{jj}^{(n)} \rightarrow 0$ as $n \rightarrow \infty$. ▪

2.5.4 Theorem:

In an irreducible Markov Chain$\{X_n, n \geq 0\}$, all the states are of same type. They are either all transient, or all persistent null, or all persistent non – null. All the states are either aperiodic or periodic and in the latter case they all have the same period say 'd'.

Proof:

Since the Markov Chain is irreducible, every state can be reached from every other state. If i,j are any two states, then i can be reached from j and j from i. That is,

$$p_{ij}^{N} = a > 0 \text{ for some } N \geq 1$$

$$p_{ji}^{M} = b > 0 \text{ for some } M \geq 1$$

And we have

$$p_{jk}^{(n+m)} = p_{jk}^{m+n} = \ {}_{r} p_{jr}^{(m)} p_{rk}^{(n)}$$

$$\geq p_{jr}^{(m)} p_{rk}^{(n)} \text{for each r.}$$

Here $$p_{ii}^{(n+N+M)} \geq p_{ij}^{(N)} p_{jj}^{(n)} p_{ji}^{(M)} = abp_{jj}^{(n)}$$

and $p_{jj}^{(n+N+M)} = p_{ji}^{(M)} p_{ii}^{\ n}\ p_{ij}^{(N)} = abp_{ii}^{(n)}$.

From the above, it is clear that the two series $_n p_{ii}^{(n)}$ and $_n p_{jj}^{(n)}$ converge or diverge together. Thus the two states i,j are either both transient or both persistent.

Suppose that i is persistent null, then $p_{ii}^{(n)} \to 0$ as n$\to \infty$, then from $p_{jj}^{(n)} \to 0$as n$\to \infty$.

So that j is also persistent null, they are both persistent null.

Suppose that i is persistent non-null and has period t, then $p_{ii}^{(n)} > 0$, whenever n is a multiple of t.

Now $$p_{ii}^{(N+M)} \geq p_{ij}^{(N)} p_{ji}^{(M)} = ab > 0.$$

So that (N+M) is multiple of t.

$$p_{jj}^{(n+N+M)} \geq abp_{ii}^{(n)} > 0.$$

Thus (n+N+M) is multiple of t and so t is the period of the state j also.

Corollary:

In a finite irreducible Markov Chain$\{X_n, n \geq 0\}$ with state space E all states are non–null persistent.

Definition:

Consider a Markov chain with transition probabilities P_{jk} and t.p.m $P = (p_{jk})$.

A probability distribution $\{v_j\}$ is called stationary (or invariant) for the given chain if $v_k = \sum v_j p_{jk}$ such that $v_j \geq 0, \sum v_j = 1$.

2.5.5 Theorem :

Let $\{X_n, n \geq 0\}$ be a Markov Chain with state space E.If state $j \in E$ is persistent, then for every state $k \in E$ that can be reached from state j, $F_{kj} = 1$.

Proof:

Let a_k be the probability that starting from state j, the state k is reached without previously returning to state j. the probability of never returning to state j once state k is reached is $(1-F_{kj})$.

The probability of the compound event that starting from state j, the system reaches state k (without returning to state j) and never returns to state j is $a_k (1-F_{kj})$. If there are some other states, say, r, s, …… then we get similar terms $a_r (1-F_{rj})$,$a_s (1-F_{sj})$. Thus the probability Q that starting from state j the system never returns to state j is given by

$$Q = a_k (1-F_{kj}) + a_r (1-F_{rj}) + a_s (1-F_{sj}) + \ldots\ldots.$$

But, the state j is persistent, $F_{jj}=1$ and the probability of never returning to state j is $1-F_{jj}=0$. Thus Q=0. This implies that each term is zero, so that $F_{kj}=1$. ■

2.5.6 Theorem: (Ergodic theorem)

For a finite irreducible, aperiodic Markov chain $\{X_n, n \geq 0\}$with t.p.m. $P = (p_{jk})$, the limit $v_k = \lim p_{jk}^{(n)}$ exists and are independent of the initial state j. the limiting probability v_k are such that $v_k \geq 0$, $\sum v_k = 1$, i.e. $\{v_k\}_{k=0}^{\infty}$ define a probability distribution.

Futher, the limiting probability distribution for the given Markov chain is $\{v_k\}_{k=0}^{\infty}$, so that $v_k = \sum v_j p_{jk}$, $\sum v_k = 1$; …. . . . . . .(1) writing $V = (v_1, v_2, \ldots, v_k, \ldots)$, $\sum v_k = 1$ the relation (1) may also be written as $V = VP$ (or) $V(P - I) = 0$.

Proof:

Since the state are aperiodic, persistent non-null, for each pair of j,k$\in$E, $\lim_{n\to\infty} p_{jk}^{(n)}$ exists and is equal to $F_{jk} \quad \mu_{kk}$.

Since k is persistent, $F_{jk}=1$, so that

$$v_k = \lim_{n\to\infty} p_{jk}^{(n)} = \frac{1}{\mu_{kk}} > 0,$$

and is independent of j.

Since $\quad {}_k\, p_{ik}^{(n)}=1, \quad {}_{k=1}^{N} p_{ik}^{(n)} \leq 1$ for all N,

$$\sum_{k=1}^{N}\left(\lim_{n\to\infty} p_{ik}^{(n)}\right) \leq 1, \text{ i.e., } \sum_{k=1}^{N} v_k \leq 1.$$

Since it holds for all $\sum_{k=1}^{\infty} v_k \leq 1$ we have $p_{jk}^{(n+m)} = \sum_i p_{ji}^{(n)} p_{ik}^{(m)}$

And $\quad v_k = \lim_{n\to\infty} p_{jk}^{(n+m)} = \lim_{n\to\infty} \sum_i p_{ji}^{(n)} p_{ik}^{(m)}$

$$\geq \sum_i \{\lim_{n\to\infty} p_{ji}^{(n)}\}\, p_{ik}^{(m)}$$

$$= \sum v_i\, p_{ik}^{(m)} \text{, for all m}$$

$$v_k \geq \sum_i v_i\, p_{ik}^{(m)}$$

Suppose, if possible, that $v_k > \sum_i v_i\, p_{ik}^{m}$ then summing overall k, we get

$$\sum_k v_k > \sum_k \sum_i v_i\, p_{ik}^{m} = \sum_i v_i,$$

which is impossible. Hence the sign of equality holds, i.e.

$$v_k = \sum_i v_i \, p_{ik}^{m} \quad \text{for all } m \geq 1.$$

In particular, when m=1, $v_k = \sum_i v_i \, p_{ik}$.

For large m, we have $v_k = \sum_i v_i v_k = \left(\sum_i v_i \right) v_k$.

Hence, $\sum v_i = 1$.

Then (v_k) defines the probability distribution.

Hence, $V = V$ P

V (P-I) = 0 = V(I-P)

This shows that v_k is a probability distribution: the distribution is unique.

The distribution is known as stationary (or equilibrium) distribution of the chain and the probabilities are known as stationary (or equilibrium) probabilities.

We state, without proof, the converse also holds.

If a chain is irreducible and aperiodic and if there exists a unique stationary distribution v_k for all chain then the chain is ergodic and $v_k = 1 \; \mu_{kk}$.

Thus ergodicity is a necessary and sufficient condition for the existence of v_k satisfying in case of an irreducible and aperiodic chain.■

2.5.7 Ergodic Theorem for Reducible Markov Chains:

Chain with a single closed class:

Let $X_n, n \geq 0$ be the finite Markov chain with aperiodic states, and tpm P.

Let P be the transition probability matrix of the m-state chain with state space S , and the transition (sub matrix) of transition among the k (≤m) members of the closed class C .

Let $V_1 = \ v_1, v_2, \ldots . v_j, \ldots v_n$ be the stationary distribution corresponding to the stochastic sub matrix P_1,i.e.$P_1^n \to nV_1$.

If $V \ = \ V_1 \ , 0$ then, as n→ $\infty, P^n \to eV$.

In other words, element wise V is the stationary distribution corresponding to the matrix P. An outline of proof is given below:

The transition probability matrix of the chain can be put in canonical form

$$P= \begin{matrix} P_1 & 0 \\ R_1 & Q \end{matrix}$$

where R the stochastic (sub) matrix corresponds to transitions among the members of class C and Q corresponds to transitions among the other states (of S-C).

We have $$P^n = \begin{matrix} P_1^n & 0 \\ R & Q^n \end{matrix} ,$$

where $R_n = R_{n-1}P_1 + Q^{n-1}R_1.$

Writing $R_1 = R$, we get

$$R_{n+1} = \ {}_{i=0}^{n} Q^i R P_1^{n-i} = \ {}_{i=0}^{n} Q^{n-i} R P_1^i.$$

As n→ ∞ $P_1^n \to nV_1$ and $Q^n \to 0.$

Again it can be shown that, as n→ $\infty, R_{n+1} \ \to nV_1$.

So that , writing $V \ = \ V_1 \ , 0$ we have $P^n \to eV$ ▪

2.6 Chapman –Kolmogorov Equation:

We have so far considered unit-step or one-step transition probabilities, the probability of X_n given X_{n-1},(i.e) the probability of the outcome at the n^{th} step or trial given the outcome at the previous step; p_{jk} gives the probability of unit-step transition from the state j at a trial to the state k at the next following trial. The m-step transition probability is denoted by

$$\Pr\ X_{m+n} = k|X_n = j\ \ = p_{jk}^{(m)}\ ,$$

where $p_{jk}^{(m)}$ gives the probability that from the state j at n[th] trial, state k is reached at (m+n)[th] trial in m-steps, (i.e) the probability of transition from the state j to the state k in exactly m steps. The number n does not occur in the r.h.s of the relation $\Pr\ X_{m+n} = k|X_n = j\ \ = p_{jk}^{(m)}$ and hence the chain is time homogeneous. The one-step transition probabilities $p_{jk}^{(1)}$ is denoted by p_{jk} for simplicity. Consider $p_{jk}^{(2)} = Pr\ X_{n+2} = k|X_n = j\ .$

The state k can be reached from the state j in two steps through some intermediate states r. Considering a fixed value of r; we have

$$\Pr\ \ X_{n+2} = k, X_{n+1} = r|X_n = j$$

$$= Pr\ X_{n+2} = k|X_{n+1} = r, X_n = j\ Pr\ X_{n+1} = r|X_n = j$$

$$= p_{rk}^{(1)} p_{jr}^{(1)} = p_{jr}p_{rk}.$$

Since these intermediate states r can assume values 1,2,3,…….., we have

$$p_{jk}^{\ 2} = Pr\ X_{n+2} = k|X_n = j\ \ = \ {}_{r}\ Pr\ X_{n+2} = k, X_{n+1} = r|X_n = j$$

$$= \ {}_{r}\, p_{jr}p_{rk}.$$

(Summing over for all the intermediate states)

By induction, we have

$$p_{jk}^{(m+1)} = Pr\ X_{n+m+1} = k|X_n = j$$

$$= \ {}_{r}\, Pr\ X_{n+m+1} = k|X_{n+m} = r\ \ Pr\ X_{n+m} = r|X_n = j$$

$$= \;{}_r\, p_{rk} p_{jr}^{(m)}$$

Similarly, we get $p_{jk}^{(m+1)} = \;{}_r\, p_{jr} p_{rk}^{(m)}$.

In general, we have

$$p_{jk}^{(m+n)} = \;{}_r\, p_{rk}^{(n)} p_{jr}^{(m)} = \;{}_r\, p_{jr}^{(n)} p_{rk}^{(m)}$$

This equation is a special case of Chapman-Kolmogorov *e*quation, which is satisfied by the transition probabilities of a Markov chain.

The equation $p_{jk}^{(m+n)} = \;{}_r\, p_{rk}^{(n)} p_{jr}^{(m)} = \;{}_r\, p_{jr}^{(n)} p_{rk}^{(m)}$ gives the inequality,

$$p_{jk}^{(m+n)} \geq p_{jr}^{(m)} p_{rk}^{(n)}, \text{ for any r.}$$

We can put the results in terms of transition matrices as follows. Let $P=(p_{jk})$ denote the unit step transition matrix of the Markov Chain and $P^{(m)} = (p_{jk}^{(m)})$ denote the m – step transition matrix of the Markov Chain transitions. For m=2, we have the matrix $P^{(2)}$ whose elements are given by

$$Pr\;\; X_{n+2} = k | X_n = j \;\; = \;{}_r\, p_{jr} p_{rk}.$$

It follows that the elements of $P^{(2)}$ are the elements of the matrix obtained by multiplying the matrix P by itself,(i.e)

$$P^{(2)} = P.P = P^2.$$

Similarly,

$$P^{(m+1)} = P^m.P = P.P^m.$$

and

$$P^{(m+n)} = P^m.P^n = P^n.P^m.$$

It should be noted that there exists non- Markovian Chains, whose transition probabilities satisfy Chapman-Kolmogorov Equation.

2.7 Determination of Higher Transition Probabilities:

Consider a Markov Chain with m states (m < ∞) having transition probability matrix (t.p.m) P= p_{ij} .

The n-step transition probabilities $p_{ij}^{(n)}$(i.e the elements of P^n) can be obtained by using the expression

$$p_{ij}^{(n)} = \quad {}_k\, p_{ik} p_{kj}^{(n-1)}, \qquad n=1,2,3,...,$$

where $p_{jj}^{(0)} = 1, p_{jk}^{(0)} = 0, \qquad k \neq j, p_{ij}^{(1)} = p_{ij}.$

We shall now show how some of the results of spectral theory of matrices could be used to determine $p_{ij}^{(n)} or P^n$more explicitly.

Assume that the eigen values t_i, i=1,2,…,m, of the stochastic matrix P are all distinct and different from zero. Suppose that (apart from multiplicative constants)

$$x_i = \quad x_{i1}, x_{i2}, \dots, x_{im} \quad ' \qquad \text{and } y_i' = (y_{i1}, y_{i2}, \dots, y_{im})$$

are respectively, the right and left-eigen vectors of P corresponding to t_i and th at X=($x_1, x_2 \dots x_m$) is the matrix formed with the eigen vectors $x_1, \dots x_m$. Suppose that D= (d_{ij}) is a diagonal matrix such that

$$d_{ii} = t_i, d_{ij} = 0, j \neq i$$

Then from the spectral theorem, we have

$$P = XDX^{-1} \qquad \text{and}$$

$$P^n = XD^nX^{-1}$$

$$= \sum_{k=1}^{m} t_k^n c_k x_k y_k' \; where \; c_k = \frac{1}{y_k' x_k}.$$

$$p_{ij}^{(n)} = \sum_{k=1}^{m} t_k^n c_k x_{ki} y_{kj} \quad ,n=0,1,2,\ldots.$$

2.8 Aperiodic Chain: Limiting Behavior:

Assume that P is primitive, i.e., the Markov Chain is aperiodic; then there will be no other eigen value with modulus equal to 1except 1. Since all the eigen values are assumed to be distinct, we may put

$$t_1 = 1, |t_i| < 1, i \neq 1.$$

Then $x_{ii} = 1$ for all i and we have

$$p_{ij}^{(n)} = c_1 y_{1j} + \sum_{k=2}^{m} t_k^n c_k x_{ki} y_{kj}.$$

Further , in the limit, as n→ ∞,

$$p_{ij}^{(n)} \to c_1 y_{1j}, \; where \; c_1 = \frac{1}{(y_1' x_1')} = \frac{1}{\sum_{j=1}^{m} x_{1j} y_{1j}},$$

i.e.,P^n tends to a matrix with all row equal to

$$c_1 y_1' = (c_1 y_{11}, c_1 y_{12}, \ldots, c_1 y_{1m}) .$$

This implies that lim $p_{ij}^{(n)}$ exists and is independent of the initial state i, if P is primitive.

Remark : To calculate this limit, when it exists, one needs to determine only the left-eigen vectors y_1'corresponding to $t_1 = 1$

2.8.1 Solution procedure for a finite state process:

Suppose the rate matrix A is given, for the Markov Process considered, the forward Kolomogrov equation $p_{ij}'(t) = \sum_k p_{ik}(t) a_{kj}$or $p'(t) = p(t)A$ together with the initial conditions $p_{ij}=\delta_{ij}$ (or P(0)=I) yield a solution $\{p_{ij}(t)\}$. We consider below a method of solution for a process with finite number of states in E. The forward equation $P'(t) = P(t)A$ has the solution

$$P(t) = P(0)\,e^{At} = e^{At}.$$

where the matrix $e^{At} = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n}$.

Assume that eigen values of A are all distinct. Then from the spectral theorem of matrices, we have $A = HDH^{-1}$,

where H is a non-singular matrix(formed with the right eigen vectors of A) and D is the diagonal matrix having for its diagonal elements d_i the eigen values of A. Now, 0 is an eigen value of A and $d_i \neq 0$, i=1,2,……… m are the other distinct eigen values, then

$$D = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & d_1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & d_m \end{pmatrix}.$$

This implies
$$D^n = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & d_1^n & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & d_m^n \end{pmatrix}$$

and $A^n = HD^nH^{-1}$.

We have
$$P(t) = I + \sum_{n=1}^{\infty} \frac{(HD^n t^n H^{-1})}{n!}$$

$$= H\left(I + \sum_{n=1}^{\infty} \frac{D^n t^n}{n!}\right) H^{-1}$$

$$=He^{Dt}H^{-1} \quad \text{where } e^{Dt} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & e^{d_1 t} & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & e^{d_m t} \end{pmatrix}.$$

The right-hand side of the above equation gives explicit solution of the matrix P(t). Note that even in the general case when the eigen values of A are not necessarily distinct, a canonical representation of $A=SZS^{-1}$ exists (Medhi, J. [46]).

2.9 Markov Decision Process

2.9.1 Introduction

The Markov Decision Process will be concerned with sequential decision making under uncertainty, which we will represent as a discrete-time stochastic process that is under the partial control of an external observer. At each time, the state occupied by the process will be observed and, based on this observation: the controller will select an action that influences the state occupied by the system at the next time point. Also, depending on the action chosen a and the state of the system s, the observer will receive a reward r(a, s)/ cost c(a, s) at each time step.

The key constituents of this model are the following:

- a set of decision epochs (time points)
- a set of system states (state space)
- a set of available actions
- the state- and action-dependent rewards or costs
- the state- and action-dependent transition probabilities on the state space.

Given such a model, we would like to know how the observer should act so as to maximize the rewards, possibly subject to some constraints on the allowed states of the system. To this end, we will be interested in finding decision rules, which specify the action be chosen in a particular epoch, as well as the policies, which consists of sequences of decision rules.

In general, a decision rule can depend not only on the current state of the system, but also on all previous states and actions. However, due to the difficulty of analyzing processes that allow arbitrarily complex dependencies between the past and the future, it is customary to focus on Markov Decision Processes (MDPs), which have the property that the future state and reward/ cost depend as the set of available actions, and the transition probabilities in each epoch on the current state.

The important questions that we will investigate are:

1. When does an optimal policy exist?
2. When does it have a particular form?
3. How can we efficiently find an optimal policy?

We will see that the choice of the optimality criterion and the form of the basic model elements has significant impact on the answers to these questions. We see these models are closed to dynamic programming models or dynamic programs.

2.9.2 Definitions and Notation

A Markov Decision Process (MDP) consists of a stochastic process along with a decision maker who observes the process and is able to select actions that influence its development over time. Along the way, the decision maker receives a series of rewards that depend on both the actions chosen and the states occupied by the process. A MDP can be characterized mathematically by a collection of five objects

$T, E, \mathrm{A}_e, p_t\ .|\ e,a\ \ r_t\ \ e,a\ : t \in T, e \in E, a \in A_e$, which are described below

1. $T = [0, \infty)$ is the set of decision epochs, which are the points in time when the external observer decides on and then executes some action. We will mainly consider processes with countably many decision epochs, in which case T is said to be discrete and we will usually take T = {1, 2,…,N} or T = {1, 2,…} depending on whether T is finite or countably infinite. Time is divided into time periods or stages in discrete problems and we assume that each decision epoch occurs at the beginning of a time period. A MDP is said to have either a finite time horizon or infinite time horizon, respectively, depending on whether the least upper bound of T (i.e., the supermum) is finite or infinite. If T = {1,2,…,N} is finite, we will stipulate that no decision is taken in the final decision epoch N.
2. E is the set of states (values) that can be assumed by the process (X_n or X_t) and is called the state space. E can be any measurable set, but we will mainly be concerned with processes that take values in state spaces that are either countable or which are compact subset of $\Re^n$.
3. For each state $e \in E, A_e$ is the set of actions that are possible when the state of the system is e. We will write $A = \bigcup_{e \in E} A_e$ for the set of all possible actions and we will usually assume that each A_e is either a countable set or a compact subset of $\Re^n$.
4. If T is discrete, then we must specify how the state of the system changes from one decision epoch to the next. Since we are interested in Markov decision processes, these changes are chosen at random from a probability distribution $p_t(j \mid e, a)$ on E that may depend on the current time t, the current state of the system “e”, and the action “a” chosen by the observer.
5. As a result of choosing action “a” when the system is in state “e” at time t, the observer receives a reward $r_t(e, a)$ which can be regarded as a profit when positive

or as a cost when negative. We assume that the rewards can be calculated, at least in principle, by the observer prior to selecting a particular action. We will also consider problems in which the reward obtained at time t can be expressed as the expected value of a function $r_t(e_t, a, e_{t+1})$ that depends on the state of the system at that time and at the next time, e.g., $r_t(e,a) = \sum_{j \in E} p_t(j \mid e,a).r_t(e,a,j)$ if E is discrete. (If E is uncountable, then we need to replace the sum by an integral and the transition probabilities by transition probability densities.) If the MDP has a finite horizon N, then since no action is taken in the last period, the value earned in this period will only depend on the final state of the system. This value will be denoted $r_N(e)$ and is sometimes called the salvage value or scrap value

Recall that a decision rule d_t tells the observer how to choose the action to be taken in a given decision epoch $t \in T$. A decision rule is said to be Markovian if it only depends on the current state of the system, i.e., d_t is a function of e_t. Otherwise, the decision rule is said to be history-dependent, in which case it may depend on the entire history of states and actions from the first decision epoch through the present. Such histories will be denoted $h_t = (e_1, a_1, e_2, a_2, e_{t-1}, a_{t-1}, et)$ and satisfy the recursion $ht = (h_{t-1}, a_{t-1}, e_t)$.

We will also write ht for the set of all possible histories up to time t. Notice that the action taken in decision epoch t not includeds ht. Decision rules can also be classified as either deterministic, in which case they prescribe a specific action to be taken, or as randomized, in which case they prescribe a probability distribution on the action set A_e and the action is chosen at random using this distribution. Combining these two classifications, there are four classes of decision rules, Markovian and Deterministic (MD), Markovian and Randomized (MR), History-dependent and Deterministic (HD), and History-dependent and Randomized (HR), and we will denote the sets of decision rules of each type available at time t by $\prod^K$, where K = MD, MR, HD, HR. In each case, a decision rule is just a function from S or H_t into A or P (A):

- $If\ d_t \in D_t^{MD}, then\, d_t : E \to A.$
- $If\ d_t \in D_t^{MR}, then\, d_t : E \to p(A).$
- $If\ d_t \in D_t^{HD}, then\, d_t : \mathrm{H}_t \to A.$
- $If\ d_t \in D_t^{HR}, then\, d_t : \mathrm{H}_t \to p(A).$

Since every Markovian decision rule is history-dependent and every deterministic rule can be regarded as a randomized rule (where the randomization is trivial), the following inclusions hold between these sets:

$$D_t^{MD} \subset D_t^{MR} \subset D_t^{HR}.$$

$$D_t^{MD} \subset D_t^{HD} \subset D_t^{HR}.$$

In particular, Markovian deterministic rules are the most specialized, where as history-dependent randomized rules are the most general.

A policy π is a sequence of decision rules d_1, d_2, d_3, for every decision epoch t = 1, 2, … and a policy is said to be Markovian or history-dependent, as well as deterministic or randomized, if the decision rules specified by the policy have the corresponding properties. We will write $\prod^K$, with K = MD, MR, HD, HR, for the sets of policies of these types. A policy is said to be stationary if the same decision rule is used in every epoch. In this case, π = (d, d, …) for some Markovian decision rule d and we denote this policy by d_∞. Stationary policies can either be deterministic or randomized and the sets of stationary policies of either type are denoted $\prod_{SD}$ or $\prod_{SR}$, respectively.

However, by exploiting the cost structure and the relative values (in terms of reward costs) several algorithms are designed to compute the optimal policy. They are

I. Linear Programming

II. Policy iteration and

III. Value iteration.

2.9.3 Linear Programming Algorithm

Step 1: Apply simplex method to compute an optimal basic solution (x_{ia}^{*}) to the following linear program:

$$Min \sum_{i \in E} \sum_{a \in A(i)} C_i(a) x_{ia}$$

subject to,

$$\sum_{a \in A(j)} x_{ja} - \sum_{i \in E} \sum_{a \in A(i)} p_{ij}(a) x_{ia} = 0, \quad \mathrm{j} \in \mathrm{E}$$

$$\sum_{i \in E} \sum_{a \in A(i)} x_{ia} = 1, \; x_{ia} \geq 0, \qquad a \in A(i) \; and \; i \in E.$$

Step 2: Start with the non-empty set $E_0 = \left\{ i \mid \sum_{a \in A(i)} x_{ia}^{*} > 0 \right\}$. For any state i$\in$ E_0, set the decision $R_i^{*} = a$ and for some ‘a’ such that $x_{ia}^{*} > 0$.

Step 3: If E_0 = E, then the algorithm is stopped with policy R^* otherwise, determine some state $i \notin E_o$ and action a $a \in A(i)$ such that $p_{ij} > 0$ some $j \in E_0$.

Next $R_i^{*} = a$ and $E_0 = E_0 \bigcup i$ repeat **Step 3.**

The linear program can heuristically be explained by interpreting the variables x_{ia}, where x_{ia} = the long-run fraction of decision epochs at which the system is in state i and action a is made.

The objective of the linear program is the minimization of the long-run average cost per time unit, while the first set of constraints represent the balance equations requiring that for any state $j \in E$ the long-run average number of transitions from state j per time unit must be equal to the long-run average number of transitions into state j per time unit. The last constraint obviously requires that the sum of the fractions x_{ia} must be equal to 1.

Two other methods namely policy and Value iteration not within the scope of this thesis.

2.9.4 Data Transformation (Semi-Markov Decision Processes)

In discrete-time Markov decision model, decisions can be made only at fixed epochs t = 0, 1, 2,.. However, in many stochastic control problems the times between the decision epochs are not constant but random. A possible tool for analyzing such problems is the semi-Markov decision model. Also, for the optimality criterion of the long-run average cost per time unit, we use a data-transformation method by which the semi-Markov decision model can be converted into an equivalent discrete-time Markov decision model. The data transformation method enables us to apply the recursive method of value-iteration to the semi-Markov decision model.

The algorithms for computing an average cost optimal policy in the discrete-time Markov decision model can be extended to the semi-Markov decision model. This can be done by the Data Transformation Technique, we discussed in the previous section. This method is an extension of transformation technique for continuous time- Markov decision processes.

First choose a number τ with, $0 < \tau \leq \min\limits_{i,a} \tau_i(a).$

Consider now the discrete-time Markov decision model whose basic elements are given by

1. $\bar{S} = S,\ i \in S$
2. $\bar{A}(i) = A(i)$
3. $C_i(a) = C_i(a) / \tau_i(a); \quad a \in A, and\ i \in S$
4. $p_{ij}(a) = \begin{cases} \tau / \tau_i(a)\ \ p_{ij}(a) & j \neq i, \quad a \in A(i)\ and\ i \in S \\ \tau / \tau_i(a)\ \ p_{ij}(a) + \left[1 - \ \tau / \tau_i(a)\ \right] & j = i, \quad a \in A(i)\ and\ i \in S \end{cases}.$

This discrete-time Markov decision model has the same class of stationary policies as the original semi-Markov decision model.

This chapter has provided some important concepts of stochastic processes in a lucid manner. With this background in Stochastic Process and steady state behavior analysis, we can study some of the multi echelon models in Supply Chain Management and Markov Decision Processes.

Chapter-3

MDP in Supply Chain: Optimal Inventory Control System

3.1 Introduction

Supply chain can be defined as the management of flow of products and services, which begins from the products point and ends at the consumption point(retailer). This process comprises of movement and storage of raw materials that are involved in work – in - progress, inventory and fully furnished goods. Supply Chain exists in both service and manufacturing organizations, but the complexity of the chain may vary greatly from industry to industry.

Inventory decision is an important component of the supply chain management, because inventories exist at each and every stage of the supply chain as raw material or semi-finished or finished goods. They can also be as work-in-process between the stages or stations. Since holding of inventories can cost anywhere between 20% to 40% of their value, their efficient management is critical in Supply Chain operations. Multi – echelon inventory control is a versatile tool to solve SCM problems.

The objective for a multi-echelon inventory model is to coordinate the inventories at various echelons so as to minimize the total cost associated with the entire multi-echelon inventory system. It might also be a suitable objective when certain echelons are managed by either the suppliers or the retailers of the company. The reason is that a key concept of supply chain management is that a company should strive to develop an informal partnership relation with its suppliers and retailers that enable them jointly to maximize their total profit.

It would be appropriate to say that information technology is a vital organ of supply chain management. With the advancement of technologies, new products are being introduced within fraction of seconds to increase their demand in the market.

Multi-echelon inventory system has been studied by many researchers and its applications in supply chain management has proved worthy in recent literature. As supply chain integrates many operators in the network and optimizes the total cost or given involved without compromising the customer service efficiency.

Continuous review models of multi-echelon inventory system in 1980's concentrated more on repairable items in a Depot-Base system than consumable items(see Graves, Moinzadeh and Lee [29], [55],). All these models deal with repairable items with batch ordering. Sven Axsäter [10] proposed an approximate model of inventory structure in SC. One of the oldest paper published in the field of continuous review multi-echelon inventory system is a basic and seminal paper written by Sherbrook in 1968 [85]. He assumed (S-1,S) polices in the Depot-Base systems for repairable items in the American Air Force and could approximate the average inventory and stock. Out level in bases. Seifbarghy and Jokar [83], analyzed a two echelon inventory system with one warehouse and multiple retailers controlled by continuous review (r,Q) policy. A ccomplete review was provided by Benita M. Beamon (1998) [12]. The supply chain concept grow largely out of two-stage multi-echelon inventory models, and it is important to note that considerable research in this area is based on the classic work of Clark and Scarf(1960) [13]. In the case of continuous review perishable inventory models with random lifetimes for the items, most of the models assume instantaneous supply of order. The assumption of positive lead times further increases the complexity of the analysis of these models and hence there are only a limited number of models dealing with positive lead-times. A continuous review perishable inventory system at Service Facilities was studied by Elango (2001) [23]. A continuous review(s,S) policy with positive lead times in two-echelon Supply Chain was studied by Krishnan, K. and Elango,C. (2006) [49]. Service facilities in the inventory in supply chain management is a quiet new area under active research.

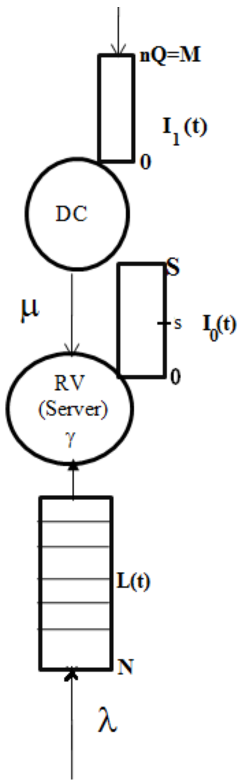
In this chapter we considered a inventory systems which is maintained in service facilities at retailer vendor in tandem supply chain, having Retailer Vendor(RV) and

Distribution Centre(DC). One item from inventory at RV is used to serve the customer, after a random service time. who . (s, S) policy is adopted at Retail Vender node for replenishment of inventory.

3.2 Model Formulation

We consider a Supply Chain system consist of Distribution Centre (DC), and Retail vendor (RV) with service facility and inventory is maintained at both DC and RV nodes. For every demand at the retailer node (RV) an item is supplied only after an exponential service time with parameter γ. The waiting space in the retailer node has maximum capacity N. An arriving customer who see N customers in the system leaves the facility. For example car service station with finite waiting space for parking.

Inventory policy adopted at RV node is (s, S) type in which order for Q=S-s>s items are placed as end, when the inventory level reaches the prefixed level s. The lead time is exponentially distributed with parameter $\mu(>0)$. Demand at RV node follows a Poisson process with parameter $\lambda(>0)$. At DC, items are packed as Q items in one pocket with maximum inventory level nQ (n pockets). The ordering policy at DC is of (0,nQ) type where the inventory level reach 0, instantaneous replenishment of nQ= M items is made. Randomized Markov Decision (MR) policy is used to solve the problem of MDP.



Fig(1) Inventory Control in Supply Chain

3.3 Analysis:

Let $I_0(t)$ and L(t) denote the inventory level and the number of customers in the system at time t in retailer node (RV) respectively. Then $\{(I_0(t), L(t)) : t \geq 0\}$ is a finite two dimensional stochastic process with state space, E = {0, 1, 2, … ,S} x {0, 1, 2, …,N}.

Here the whole Supply Chain Inventory Management (SCIM) is observed and controlled by an agent or manager through complete internet based information systems.

Some of the state transitions in the Markov Process $\{(I_0(t), L(t)) : t \geq 0\}$ with the corresponding rate of transitions are given below

(i) The arrival of a demand for an item at retailer node (RV) makes a state transition from (i,q) to (i,q+1) with rate λ.

(ii) Replenishment of inventory at retailer node, with order quantity Q makes a state transition from (i,q) to (i+Q,q) with rate μ>0.

(iii) The transition from state (i,q) to (i-1, q-1) occurs with rate γ.

3.4 MDP Formulation

Decision epochs: We choose the decision epochs of the MDP as the time points at which service completion occurs.

State Space: $E = \ 0, 1, 2, \ldots, S \ \times \ 0, 1, 2, \ldots, N \ .$

Action Set: The reordering decisions (0- no order;1- order; 2 –compulsory order) taken at each state of the system (i, q) $\in$ E when $i \le s$, where s a prefix reorder level such that $S - i > s$. The compulsory order for S items is made when inventory level is zero. The set of possible actions are $A_1 = \{0\}$, $A_2 = \{0, 1\}$, $A_3 = \{2\}$, then

$$A_{E_k} = \begin{cases} \{0\}, & s+1 \le i \le S,\ 0 \le \mathrm{q} \le \mathrm{N}, \\ \{0,1\}, & 1 \le i \le s,\ 0 \le \mathrm{q} \le \mathrm{N}, \\ \{2\}, & i = 0,\ 0 \le \mathrm{q} \le \mathrm{N}. \end{cases}$$

$A = A_1 \bigcup A_2 \bigcup A_3 = \bigcup_{(\mathrm{i},q)\in E} A_{(\mathrm{i},q)}$, the corresponding state space classes are given by

$$E = E_1 \bigcup E_2 \bigcup E_3,$$

$$E_1 = \ (i,q) : s+1 \le i \le S,\ 0 \le \mathrm{q} \le \mathrm{N} \ ,$$

$$E_2 = \ (i,q) : 1 \le i \le s,\ 0 \le \mathrm{q} \le \mathrm{N} \ ,$$

$$E_3 = \ (i,q) : i = 0,\ 0 \le \mathrm{q} \le \mathrm{N} \ .$$

Suppose the policy f (sequence of decisions) is defined as a function f : E → A, given by

$$f(i,q) = \ a : (i,q) \in E,\ a \in A_m,\ m = 1,2,3 \ .$$

Transition probability: $p_{(i,q)}^{(j,r)}(\mathrm{a})$ denote the transition probability from state (i,q) to state (j,r) when decision *a* is made at state (i,q).

Cost: *C((i,q),a)* denote the cost occurred in the system when action *a* is taken at state (i,q).

3.5 Steady State Analysis

Let R denote the stationary policy, which is randomized time invariant Markovian Policy (MR). From our assumptions it can be seen that $(I_0 \ t \ , L \ t \ ) : t \ge 0$

become as the controlled process $\{(I_0^R(t), L^R(t)) : t \geq 0\}$ when policy R is adopted. Since the process $\{(I_0^R(t), L^R(t)) : t \geq 0\}$ is a Markov Process with finite state space E, it is completely Ergodic, if every stationary policy gives an irreducible Markov chain. It can be seen that for every stationary policy $f \in F$, (I_0^f, L^f) is completely Ergodic and also the optimal stationary policy R^* exists, because the state and action spaces are finite.

If d_t is the Markovian randomized (MR) decision rule, the expected reward satisfies the transition probability relations.

$$p_t\left((j,r) \mid (i,q), d_t(i,q)\right) = \sum_{a \in A_s} p_t\left((j,\mathrm{r}) \mid (\mathrm{i,q}), \mathrm{a}\right) p_{\mathrm{d}_t(i,q)}(a).$$

$$r_t\left((i,q), d_t(i,\mathrm{q})\right) = \sum_{a \in A_s} r_t(\mathrm{i,q,a}) p_{\mathrm{d}_t(i,q)}(a).$$

For randomized Markovian Policy $f \in F^{MR}$ where, F^{MR} denotes the randomized Markovian (MR) policy. Under this policy F an action $a \in A$ is chosen with probability $F_a(i,q)$, whenever the process is in state $(i,q) \in E$. Whenever $F_a(i,q)= 0$ or 1, the stationary Markovian policy F reduces to a familiar stationary policy.

State	s	s-1	…	1
Order size	Q=S-s	Q+1=S-s+1	…	Q+s-1=S-1
Probability	p_s	p_{s-1}	…	P_1

Under the randomized policy f, the expected long run total cost rate is given by

$$C^f = h\bar{I}^f + c_1 w^f + c_2 \eta_a^f + g\eta_b^f + \beta\eta_c^f, \tag{1}$$

h - holding cost / unit item / unit time,

c_1 – waiting cost / customer / unit time,

c_2 – reordering cost / order,

g - balking cost / customer,

β - service cost / customer,

$\bar{I}^f$ - mean inventory level,

$\bar{w}^f$ - mean waiting time,

η_a^f - reordering rate,

$\eta_b{}^f$ - balking rate,

$\eta_c{}^f$ - service completion rate.

Our objective is to find an optimal policy f^* for which $C^{f^*} \leq C^f$ for every MR policy in F^{MR}

For any fixed MR policy $f \in F^{MR}$ and $(i,q),(j,r) \in E,$ define

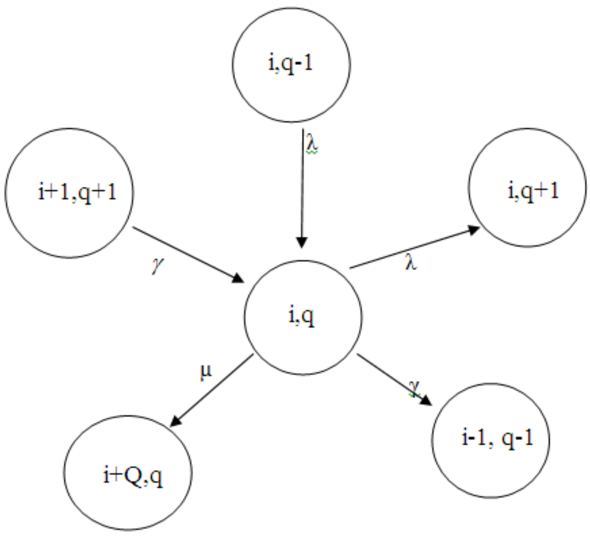
$$\Phi_{iq}^f(j,r,\text{t}) = P\text{r}\ \ I_0{}^f(t) = j, \text{L}^f(t) = \text{r} \mid I_0{}^f(0) = \text{i}, \text{L}^f(0) = q \ \ , \ (i,q),(j,r) \in E. \quad (2)$$

Now $\Phi_{iq}^f \ \ j,r,t$ satisfies the Kolmogorov - forward differential equation $\Phi_i{}'(\text{t}) = \Phi(\text{t})\text{A},$ where A is an infinitesimal generator of the Markov process $\{(\text{I}_0{}^f(\text{t}), L^f(\text{t})) : \text{t} \geq 0\}.$

For each MR policy f, we get an irreducible Markov chain with the state space E and actions space A which are finite.

Clearly $\Phi^f(\text{j, r}) = \lim_{t\to\infty} \Phi_{iq}^f \ \ j,r;t$ exists and is independent of initial state conditions.

Fig (2) represent the in-rate and out-rate flow diagram of the system states.



fig(2)

Now the system of equations can be written in order as follows,

$$\lambda\Phi^{f}(S,0)=\mu\sum_{j=0}^{S}p_j\Phi^{f}(j,0) \tag{3}$$

$$(\lambda+\gamma)\Phi^{f}(S,r)=\mu\sum_{j=0}^{S}p_j\Phi^{f}(j,r)+\lambda\Phi^{f}(S,r-1),\quad 1\le r\le M-1 \tag{4}$$

$$\gamma\,\Phi^{f}(S,N)=\mu\sum_{j=0}^{S}p_j\Phi^{f}(j,N)+\lambda\Phi^{f}(S,N-1) \tag{5}$$

$$\lambda\Phi^{f}(j,0)=\gamma\Phi^{f}(j+1,1),\quad s+1\le j\le S-1 \tag{6}$$

$$(\lambda+\gamma)\Phi^{f}(j,r)=\gamma\,\Phi^{f}(j+1,r+1)+\lambda\Phi^{f}(j,r-1),\ s+1\le j\le S-1;1\le r\le N-1 \tag{7}$$

$$\gamma\,\Phi^{f}(j,N)=\lambda\Phi^{f}(j,N-1),\quad s+1\le j\le S-1 \tag{8}$$

$$(\lambda+\mu p_j)\Phi^{f}(j,0)=\gamma\,\Phi^{f}(j+1,1)\ ,\ 1\le j\le s \tag{9}$$

$$(\lambda+\mu p_j+\gamma)\Phi^{f}(j,r)=\gamma\Phi^{f}(j+1,r+1)+\lambda\Phi^{f}(j,r-1), \tag{10}$$

$1\le j\le s;1\le r\le N-1$,

$$(\mu p_j+\gamma)\Phi^{f}(j,N)=\gamma\Phi^{f}(j,N-1),1\le j\le s\ , \tag{11}$$

$$(\lambda+\mu p_0)\Phi^{f}(0,0)=\gamma\,\Phi^{f}(1,1)\ , \tag{12}$$

$$(\lambda+\mu p_0)\Phi^{f}(0,r)=\gamma\,\Phi^{f}(1,r+1)+\lambda\Phi^{f}(0,r-1),\quad 1\le r\le N-1\ , \tag{13}$$

$$\mu p_0\Phi^{f}(0,N)=\lambda\Phi^{f}(0,N-1) \tag{14}$$

The above set of equations together with the condition $\sum_{(j,r)\in E}\Phi^{f}(j,r)=1$

,determine the steady-state probabilities uniquely. (15)

3.6 System Performance Measures

Since the probability Φ^{f}(,j r) also gives the value that the long-run fraction of time the system is in the state (j, r), we derive the following system performance measures.

1. The expected inventory level in the system is given by

$$\overline{I^{f}}=\sum_{j=1}^{S}\sum_{r=0}^{N}j\Phi^{f}(j,r). \tag{16}$$

2. The mean waiting time is given by

$$\overline{W}^{\,f} = \sum_{r=1}^{N} \frac{r}{\gamma} \sum_{j=0}^{S} \Phi^f(j,r) + \sum_{k=0}^{s} \frac{1}{\mu p_k} \sum_{m=1}^{[N/S]} \sum_{r=1}^{mS} m\Phi^f(j,r). \tag{17}$$

3. The reorder rate is given by

$$\eta_a{}^f = \gamma \sum_{r=0}^{N} \sum_{j=0}^{s} p_j \Phi^f(j,r). \tag{18}$$

4. The balking rate is given by

$$\eta_b{}^f = \lambda \sum_{j=0}^{S} \Phi^f(j,N). \tag{19}$$

5. The service completion rate is given by

$$\eta_c{}^f = \gamma \sum_{r=1}^{N} \sum_{j=1}^{S} \Phi^f(j,r). \tag{20}$$

Hence the average cost rate of the system is given by

$$C^f = h\sum_{j=1}^{S} j \sum_{r=0}^{N} \Phi^f(j,r) + \frac{c_1}{\gamma} \sum_{r=1}^{N} r \sum_{j=0}^{s} \Phi^f(j,r) + \frac{c_1}{\mu} \sum_{m=1}^{[N/S]} \sum_{r=1}^{mS} \sum_{j=0}^{S} \frac{m}{p_j} \Phi^f(j,r) + c_2\gamma \sum_{r=1}^{N} r \sum_{j=1}^{s+1} p_j \Phi^f(j,r)$$
$$+ g\lambda \sum_{j=0}^{S} \Phi^f(j,\mathrm{N}) + \beta\gamma \sum_{j=1}^{S} \sum_{r=1}^{N} \Phi^f(j,r).$$

(21)

3.7 Linear programming problem

3.7.1 LPP Formulation

In this section we propose a LPP model within a MDP framework. First define the variables, D (j, r, k) as a conditional probability such that

D(j, r, k) = Pr {decision is k | state is (j, r)}. (22)

Since $0 \leq$ D (j, r, k) ≤ 1, this is compatible with the randomized time invariant Markovian policies. Here, the Semi – Markovian decision problem can be formulated as a linear programming problem.

Hence $0 \leq D(j, r, k) \leq 1$ and $\sum_{k \in A=\{0,1,2\}} D(\mathrm{j,r,k}) = 1, 0 \leq \mathrm{r} \leq \mathrm{N};\ 0 \leq \mathrm{j} \leq \mathrm{S}.$

For the reformulation of the MDP as LPP, we define another variable y (j, r, k) as follows:

$$y(\mathrm{j,r,k}) = \mathrm{D(j,r,k)}\Phi^{f}(\mathrm{j,r}). \tag{23}$$

From the above definition of the transition probabilities

$$\Phi^{f}(\mathrm{j,r}) = \sum_{k\in A} y(\mathrm{j,r,k}),\ (\mathrm{j,r}) \in \mathrm{E},\ \mathrm{k} \in A = \{0,1,2\}. \tag{24}$$

Expressing $P^{\pi}(\mathrm{j,r})$ in terms of $y(\mathrm{j,r,k})$, the expected total cost rate function (21) is given by

Minimize

$$\begin{aligned} C = {} & h\sum_{k\in A}\sum_{j=1}^{S} j\sum_{r=1}^{N}\mathrm{y(j,r,k)} + h\sum_{j=1}^{S} j.y(\mathrm{j,0,0}) + c_1\gamma\sum_{k\in A}\sum_{r=1}^{N}\sum_{j=0}^{s} p_j.y(\mathrm{j,r,k}) \\ & + c_2\sum_{k\in A}\sum_{r=1}^{N}\frac{r}{\gamma}\sum_{j=0}^{S}\mathrm{y(j,r,k)} + \frac{c_2}{\mu}\sum_{k\in A}\sum_{m=1}^{[N/s]}\sum_{r=1}^{ms}\sum_{j=0}^{S}\left(\frac{m}{p_k}\right)\mathrm{y(j,r,k)} \\ & + g\lambda\sum_{k\in A}\sum_{j=1}^{S}\mathrm{y(j,N,k)} + g\ \lambda\ y(0,\mathrm{N},0) + \sum_{k\in A}\sum_{j=1}^{S}\sum_{r=1}^{N}\ \beta_k y(\mathrm{j,r,k}), \end{aligned} \tag{25}$$

subject to the constraints,

(1) $y(\mathrm{j,r,k}) \geq 0,\ (\mathrm{j,r}) \in \mathrm{E},\ \mathrm{k} \in \mathrm{A}_l,\ l = 1,2$,

(2) $\sum_{l=1}^{2}\sum_{(\mathrm{j,r})\in \mathrm{E}_l}\sum_{k\in A_l} y(\mathrm{j,r,k}) = 1.$

The balance equations (3) – (14) are obtained by replacing $\Phi^{\pi}(\mathrm{j,r})$ by $\sum_{k\in A} y(\mathrm{j,r,k})$.

<u>3.7.1 Lemma:</u>

The optimal solution of the above Linear Programming Problem yields a deterministic policy.

<u>Proof:</u>

From the equations

$$y(\mathrm{j,r,k}) = \mathrm{D(j,r,k)}\Phi^{\pi}(\mathrm{j,r}) \tag{26}$$

and

$$\Phi^{\pi}(\mathrm{j,r}) = \sum_{k\in A} y(\mathrm{j,r,k}),\ (\mathrm{j,r}) \in \mathrm{E}. \tag{27}$$

Since the decision problem is completely ergodic every basic feasible solution to the above linear programming problem has the property that for each $(\mathrm{j}, \mathrm{r}) \in \mathrm{E}, y(\mathrm{j}, \mathrm{r}, \mathrm{k}) > 0$ for every k∈A.

3.8 Numerical Illustration and Discussion

In this system we consider a problem to illustrate the method described in section 4, through numerical examples. We implemented TORA software to solve LPP by simplex algorithm. In all cases, the computational time is considerably less in a Intel 2000 machine. For numerical illustration, we consider the sub - problem of our main which assumes instantaneous replenishment. That is lead time is zero and the service rate is exponentially distributed with parameter γ. In this case ,we can get the expected cost rate C(γ) which can be found from the previous results obtained by Sapna, K. P., and Berman,O. [81]. The objective is to prove the conjecture that the proper service rate be employed to optimize the cost and efficiency of the service facility depends only on the number of the customers in the system and not on the inventory level. Now the expected cost rate when lead time is zero ($\frac{1}{\mu}=\infty$) is given by

$$C(\gamma) = h\left(\frac{S+1}{2}\right) + \frac{c_1}{s\left[m + \frac{\alpha(1)}{\lambda}\right]} + c_2 \sum_{k\in\{0,1,2\}} \sum_{j=1}^{s} k\ m\ \Phi^f(\mathrm{j},\mathrm{k}) + g\sum_{j=0}^{s} \Phi^f(\mathrm{j},\mathrm{N}) + \beta\gamma\sum_{j=1}^{s}\sum_{r=1}^{N}\Phi^f(\mathrm{j},\mathrm{r})$$

where $m = \frac{1}{\gamma}, \quad \alpha(1) = \dfrac{1-\frac{\lambda}{\gamma}}{1-\left(\frac{\lambda}{\gamma}\right)^N}.$

For $0 \le \mathrm{j} \le \mathrm{S},\ 1 \le \mathrm{r} \le \mathrm{N}$

$$\Phi^f(\mathrm{j},\mathrm{r}) = \left(\frac{\lambda}{\gamma}\right)^{r-1} \Phi^f(\mathrm{j},0).$$

For $0 \le j \le S,\ \Phi^{f}(j,0) = \left(\frac{1}{S}\right)\left(\frac{1-\frac{\lambda}{\gamma}}{1-\left(\frac{\lambda}{\gamma}\right)^{N}}\right)$,

Here β denotes the cost associated with the service rate γ. Since C(γ) is the univariate function of γ, it can be differential with respect to γ, and the minimum value of C(γ) can be obtained for specific value γ*. When different service rates are used at different states of the systems, we conclude the conjecture easily.

By studying many number of numerical examples(by computer programming) be found that service rate is insensitive to changes in the inventory level. Also the service rates are independent of ordering cost(c_2) , inventory carrying cost(h) and balking cost (g). But the following parameters are sensitive to the service rates (i) arrival rate (ii) waiting time cost (c_1) and (iii) number of customer in the system(r).

Typical numerical results are summarized in the following Tables 1-4 where 'r' is the number of the customers in the system [(j,r) denote the system state where' j' denotes the inventory level and 'r' denotes the number of customers]

The first two tables represent the effect of changes in waiting time costs and the next two show effect in the arrival rate. We take $\gamma_1 = 1, \gamma_2 = 2, \gamma_3 = 3, \gamma_4 = 4$.

Objectives:

- **(i)** As the waiting cost c_1 increases, larger service rates are utilized for larger values of r. For example in Table 1,when c_1=1,γ_1 =1 is used for all r values for r=9,10, γ_2=2 is used. When c_1=10,γ_1 =1 is not used at all but γ_2 , γ_3, γ_4 are used.

- **(ii)** As the arrival rate λ is increasing, larger service rates are utilized for larger values of n.

Tables 1

Optimal rates for c_2=25,h=0.2,g=2, $\beta_1 = 1, \beta_2 = 3, \beta_3 = 4, \beta_4 = 5$,S=20, s=7, N=15, λ=8, μ=0.4

c_1	γ_1=1	γ_2=2	γ_3=3	γ_4=4
1	$1 \le r \le 8$	$9 \le r \le 10$	_	_
2	$1 \le r \le 7$	$8 \le r \le 10$	_	_
3	$1 \le r \le 6$	$7 \le r \le 10$	_	_
4	$1 \le r \le 5$	$6 \le r \le 9$	r=9	_
5	$1 \le r \le 4$	$5 \le r \le 8$	$9 \le r \le 10$	_
6	$1 \le r \le 3$	$4 \le r \le 7$	$8 \le r \le 9$	r=10

Tables 2

Optimal rates for c_2=25,h=0.2,g=3, $\beta_1 = 1, \beta_2 = 4, \beta_3 = 6, \beta_4 = 8$, S=20, s=7, N=15, λ=8, μ=0.4

c_1	γ_1=1	γ_2=2	γ_3=3	γ_4=4
1	$1 \le r \le 9$	$r = 10$	_	_
2	$1 \le r \le 8$	$9 \le r \le 10$	_	_
3	$1 \le r \le 7$	$8 \le r \le 9$	r=10	_
4	$1 \le r \le 6$	$7 \le r \le 8$	$9 \le r \le 10$	_
5	$1 \le r \le 4$	$5 \le r \le 6$	$7 \le r \le 10$	_
6	$1 \le r \le 4$	$5 \le r \le 6$	$7 \le r \le 9$	r=10

In Table1, the costs associated with the different rates are linear functions of the rates(β_i=γ_i and β_i=3γ_i respectively).

In Table 2, the cost increase for changing the rate from γ_1 toγ_2 (β_1=γ_1=1, β_2=4), the number of states for which the rate is γ_2 is smaller than that in table 1.

Tables 3

Optimal rates for c_1=20, c_2=25,h=0.2,g=3, $\beta_1 = 1, \beta_2 = 3, \beta_3 = 4, \beta_4 = 5$,S=20 , s=7 , N=15, λ=8, μ=0.4

Λ	γ_1=1	γ_2=2	γ_3=3	γ_4=4
1	$1\le r\le 7$	$8\le r\le 10$	_	_
2	$1\le r\le 6$	$7\le r\le 10$	_	_
3	$1\le r\le 6$	$7\le r\le 10$	_	_
4	$1\le r\le 5$	$6\le r\le 10$	_	_
5	$1\le r\le 4$	$5\le r\le 9$	$r=10$	_
6	$1\le r\le 4$	$5\le r\le 9$	$r=10$	_

Tables 4

Optimal rates for c_1=20, c_2=25,h=0.2,g=3, $\beta_1=1,\beta_2=4,\beta_3=6,\beta_4=8$, S=20, s=7, N=15, λ=8, μ=0.4

λ	γ_1=1	γ_2=2	γ_3=3	γ_4=4
1	$1\le r\le 8$	$9\le r\le 10$	_	_
2	$1\le r\le 8$	$9\le r\le 10$	_	_
3	$1\le r\le 7$	$8\le r\le 9$	r=10	_
4	$1\le r\le 6$	$7\le r\le 9$	r=10	_
5	$1\le r\le 7$	$6\le r\le 8$	$9\le r\le 10$	_
6	$1\le r\le 7$	$6\le r\le 8$	$9\le r\le 10$	_

Tables 3 and 4 are similar to Tables 1 and 2, but we vary λ for a fixed value c_1=20.

The case where γ_n=nγ(n is an integer),can be interpreted as the service efficiency is increased by increasing the number of servers say n.

3.9 Conclusions and future research

In this model we use semi-MDP as a tool to find optimal ordering policy at the vendor node. Linear programming technique is used to determine the optimal reorder level policy that smoothen the implementation of inventory control supply chain. The MDP considered in this model uses randomized Markov policy, which is first time introduced in this model. Implementation of history dependent policy (HR, HD) may be tried in future models.

Chapter -4

Perishable Inventory Control in Supply Chain : A Semi-MDP Model

4.1 Introduction

In this chapter we introduce a new Semi – MDP model for service facility with perishable inventory. Supply chain is a network of facilities and distribution options that performs the functions of procurement of materials, transformation of these materials into intermediate and finished products and the distribution of these finished products to customers. Supply Chain intogether is concerned with functional amolgenaration and coordination among partners. In an independently managed SC each member act independently and this to maximize his gain, by optimizing the optimal costs.

A Markov decision process is a sequential decision – making stochastic process characterized by five elements: Decision Epochs (T), States (E), Action set (A), transition probabilities ((p(.|a)) and rewards r(.|a). the 5 tuple (T,E,A,p(.|a),r(.|a)) represent the MDP.

Inventory decision is an important component of the supply chain management, because inventories exist at each and every stage of the supply chain as raw material or semi-finished or finished goods. Further maintaining perishable category of inventory in a supply chain is a tedious task rather than the stock with full life time. .They can also be as work-in-progress between the stages or stations and perishable during this time period. Since holding of inventories can cost anywhere between 20% to 40% of their value, their efficient management is critical in Supply Chain operations

Supply Chain Management (SCM) is an essential element to operational efficiency. SCM strategy can be applied to customer satisfaction and company success with social settings. In Supply Chain Management a certain points in time an agent intervenes and take decisions to correct the future path of systems. At each decision epoch the service facility system in turn the supply chain occupies a decision making state. As a converge of a right decision at a system, the agent (manager) receives a reward (incur a cost) and the system goes to the next state with a certain probability say transition probability. In this model we use long run excepted cost rate to get the optimal reordering policy (set of decisions at each state of the system)

Information technology has a substantial impact on supply chains. Scanners collect sales data at the point-of-sale, and Electronic Data Interchange (EDI) allows these data to be shared immediately with all stages of the supply chain. This technology has simplified the task of maintaining perishable inventory in SC because of instantaneous removal of perished item from stock is possible.

Multi-echelon inventory system has been studied by many researchers and its applications in supply chain management has proved worthy in recent literature. As supply chains integrates many operators in the network and optimize the total cost involved without compromising as customer service efficiency. Even though studies related to perishable inventory in SCM is very rare.

Continuous review models of multi-echelon inventory system in 1980's concentrated more on repairable items in a Depot-Base system than as consumable items[see Graves, Moinzadeh and Lee [29,55]). All these papers deal with repairable items with batch ordering. Seifbarghy and Jokar [83] analyzed a two echelon inventory system with one warehouse and multiple retailers controlled by continuous review (R,Q) policy. A Complete review was provided by Benita M. Beamon (1998) [12]. the supply chain concept grow largely out of two-stage multi-echelon inventory models, and it is important to note that considerable research in this area is based on the classic work of Clark and Scarf(1960) [18]. In the case of continuous review perishable inventory models with random lifetimes for the items, most of the models assume instantaneous supply of order. The assumption of positive lead times further increases the complexity of the analysis of these models and hence there are only a limited number of papers dealing with positive lead-times(n). A continuous review perishable inventory system at Service Facilities with zero lead time was studied by Elango, C. (2001) [21]. A continuous review(s,S) policy with positive lead times in two-echelon Supply Chain was considered by Krishnan, K., and Elango, C. [49].

In this chapter we considered a inventory system maintained in a service facilities at Retailer vendor in tandem supply chain having Retailer vendor (RV) and Distribution Centre(DC). One item (perishable) from inventory at RV is used to serve the customer. (s, S) policy is adopted at Retail Vender node. Reordering decision taken different service completion epochs. This MDP is solved using LPP technique. Numerical examples with several instances are provided to prove as conjecture that the service rate employed is depending on the number of customer in the system not on the inventory level.

4.2 Model Formulation

We consider a Supply Chain system consist of Distribution Centre (DC), Retail vendor(RV) with service facility and inventory is maintained at both DC and RV nodes. For every demand at the retailer node (RV) an item is supplied only after a exponential service time with parameter γ. The waiting space in the retailer node has maximum capacity N. An arriving customer seeing N customers in the system leaves the service area immediately.

Inventory policy adopted at RV node is (s, S) type in which order for Q=S-s>s items are placed when the inventory level reaches the prefixed level s, and lead time is exponentially distributed with parameter $\mu(>0)$. Demand at RV node follows a Poisson process with parameter $\lambda(>0)$. At DC, items are packed as Q items in one pocket with maximum inventory level nQ (n pockets). The ordering policy at DC is of (0,nQ) type where the inventory level reach 0, instantaneous replenishment M=nQ items is made. . Each item in inventory at RV has the perishable rate θ(>0), that is exponentially distributed life time having parameter Q>0.

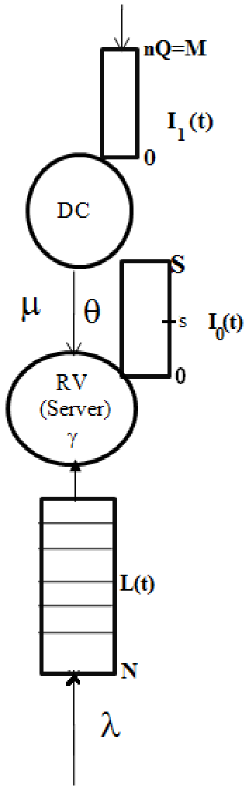


fig (1) Inventory Control in Supply Chain

4.3 Analysis:

Let I(t) and X(t) denote the inventory level and the number of customers in the system at time t. Then $\{(I(t), X(t)) : t \geq 0\}$ is a two dimensional Markov process with state space , $E_1 \times E_2$, where E_1 = {0, 1, 2, … ,S} and E_2 = {0, 1, 2, …,N}.

Some of the state transitions with the corresponding rate of transitions are given below:

(i) A transition from (i,q) to (i,q+1) takes place when an arrival of a demand at retailer node for which the rate of transition is λ.

(ii) Replenishment of inventory at retailer node makes a state transition from (i,q) to (i+Q,q) with rate μ>0.

(iii) A transition from (i,q) to (i-1, q-1) takes place after a service completion with rate γ.

4.4 MDP Formulation

Decision epochs: The service completion points perishing items and inventory items (random time points) on time lines be taken as decision epochs, as change in inventory level occurs at these points.

State Space:

E = {0, 1, 2, … ,S} X {0, 1, 2, …,N}.

Action Set: The reordering decisions (0- no order;1- order; 2 –compulsory order) taken at each state of the system (i, q) $\in$ E. The compulsory order for S items is made when inventory level is zero. The set of possible actions are $A_1 = \{0\}$, $A_2 = \{0, 1\}$, $A_3 = \{2\}$ and $A = A_1 \cup A_2 \cup A_3$.

$$A = \begin{cases} \{0\}, & s+1 \le j \le S,\ 0 \le \mathrm{q} \le \mathrm{N} \\ \{0,1\}, & 1 \le j \le s,\ 0 \le \mathrm{q} \le \mathrm{N} \\ \{2\}, & j = 0,\ 0 \le \mathrm{q} \le \mathrm{N} \end{cases}, \quad A = \bigcup_{j \in E} A_j .$$

Suppose denote the class of all stationary policies, then a policy *f* (sequence of decisions) can be defined as a function f : E $\rightarrow$ A, given by

$$f(i,q) = \begin{cases} \{0,1\} & 1 \le i \le s, q \in E_2 \\ \{0\} & s+1 \le i \le S \quad , q \in E_2 \\ \{2\} & i = 0, q \in E_2 \end{cases}$$

Let E_1 = {(i, q) $\in$ E / *f* (i, q)= 0}.

E_2 = {(i, q) $\in$ E / *f* (i, q)= 0 or 1}.

E_3 = {(0, q) $\in$ E / *f* (i, q)= 2}, 0 represents ‘no order’, 1 means reorder for ‘S-i’ items at level i and 2 means compulsory order for S items when inventory level is zero.

Transition Probability: $p_{(j,q)}^{(k,r)}(a)$ denote the transition probability from state (j, q) to state (k, r) when decision a is made at state (j, q).

Let R denote the stationary policy, which is randomized time invariant and Markovian Policy (MR). From our assumptions it can be seen that $(I(t), X(t)) : t \geq 0$ is denoted as the controlled process $\{I^R(t), X^R(t) : t \geq 0\}$ when policy R is adopted. Since the process $\{I^R(t), X^R(t) : t \geq 0\}$ is a Markov Process with finite state space E. The process is completely Ergodic, if every stationary policy gives rise to an irreducible Markov chain. It can be seen that for every stationary policy $f \in F$, $\{I^f, X^f\}$ is completely Ergodic and also the optimal stationary policy R^* exists, because the state and action spaces are finite.

A randomized Markov decision rule from the class F is equivalent to the function $f : E \to A$ given by $P_{d_t} \in \wp(A_r), j \in E_1,$ where d_t is the Markovian randomized decision rule for t ∈ T. We denote the set of decision rules at time t by D_t^{MR}.

If d_t is the Markovian randomized decision rule, the expected reward satisfies the transition probability relations.

$$p_t\left((j,r) \mid (i,q), d_t(i,q)\right) = \sum_{a \in A_s} p_t\left((j,r) \mid (i,q), a\right) p_{d_t(i,q)}(a).$$

$$r_t(i,q), d_t(i,q) = \sum_{a \in A_s} r_t(i,q,a) p_{d_t(i,q)}(a).$$

For Markovian $f \in f^{MR}$, d_t depends on history analysis through the current state of the process $(i,q) \in E$ so that $p^f\{Y_t = a \mid Z_t = h_t\} = P_{d_t(h_t)}(a)$ where Y_t – denote the action at time t and the history process Z_t defined by $Z_1(w)=s_1$ and $Z_t(w)= \{s_1,s_2,s_3,\ldots,s_t\}$ for $1 \leq t \leq N, N \leq \infty$

Randomized Markovian Policy f

Order size	Q=S-s	Q+1=S-s+1	…	Q+s=S

Probability	p_s	p_{s-1}	…	p_0

f^{MR} is the randomized Markovian policy. Under this policy f an action $a \in A(j)$ is chosen with probability $f_a(j)$, whenever the process is in state $j \in E$.

Whenever $f_a(j) = 0 or 1$, the stationary randomized policy f reduces to a familiar stationary policy.

Objective of the problem is to find the optimal reorder level s so that the long run expected total cost rate is minimum.

Notations and Assumptions:

1. $E_1 \times E_2 = E$ *is the state space of the Stochastic Process* $\{I(t), X(t) : t \geq 0\}$,

   where $E_1 = \{0,1,2,,...,S\}$ *and* $E_2 = \{0,1,2,,...,N\}$
2. $A_{(i,q)}$ – *decision set corresponding to state* $(i,q) \in E$.
3. $C_{(i,q)}(a)$ – cost *occured when action a is taken at state* (i,q).
4. $p_{(i,q)}^{(j,r)}(a)$ – *the transition probability from state* (i,q) *to state* (j,r).

   when action a is taken at state $(i,q) \in E$.
5. *Inventory levels are reviwed at the time of service completion epoch.*
6. Reordering policy is (s, S): Q = S-i > s ($1 \leq i \leq s$) items ordered when the inventory level reaches s (prefixed level), where $0 \leq s \leq S$.
7. F- the class of stationary policies.

4.5 Steady State Analysis

Let $\{I^R(t), X^R(t) : t \geq 0\}$ denote the stochastic process $\{I(t), X(t) : t \geq 0\}$, In which R is the policy adopted from our assumptions made in the previous section. The controlled process $\{I^R, X^R\}$, where R is the randomized Markovian policy in a Markov process. Under the randomized policy, f the expected long run total cost rate when policy f is adopted is given by

$$C^{f} = h\bar{I}^{f} + c_1 w^{f} + c_2 \alpha_a^{f} + g\alpha_b^{f} + \beta\alpha_c^{f} \qquad (1)$$

h - holding cost / unit item / unit time,

c_1 – waiting cost / customer / unit time,

c_2 – reordering cost / order,

g - balking cost / customer,

β - service cost / customer,

$\bar{I}^{f}$ - mean inventory level,

$\bar{w}^{f}$ - mean waiting time in system,

α_a^{f} - reordering rate,

$\alpha_b{}^{f}$ - balking rate,

$\alpha_c{}^{f}$ - service completion rate,

$\alpha_d{}^{f}$ - expected perishing rate.

Our objective is to find an optimal policy f^{*} for which $C^{f^*} \le C^{f}$ for every MR policy in F^{MR}.

For any fixed MR policy $f \in F^{MR}$ and $(i,q),(j,r) \in E,$ define

$$\Pi_{iq}^{f}(j,r,\mathrm{t}) = P\mathrm{r} \;\; I^{f}(t) = j, \mathrm{L}^{f}(t) = \mathrm{r} \mid I^{f}(0) = \mathrm{i}, \mathrm{L}^{f}(0) = q \;\;, (i,q),(j,r) \in E. \qquad (2)$$

Now $\Pi_{iq}^{f} \;\; j,r,t$ satisfies the Kolmogorov forward differential equation

$\Pi_i\,'(\mathrm{t}) = \Pi(\mathrm{t})\,\mathrm{A},$ where A is an infinitesimal generator of the Markov process $\{(\mathrm{I}^{f}(\mathrm{t}), \mathrm{X}^{f}(\mathrm{t})) : \mathrm{t} \ge 0\}.$

For each MR policy f, we get an irreducible Markov chain with the state space E and actions space A which are finite.

Clearly Π^{f} (j, r) = $\lim_{t\to\infty} \Pi_{iq}^{f} \; j,r;t$ exists and is independent of initial state.

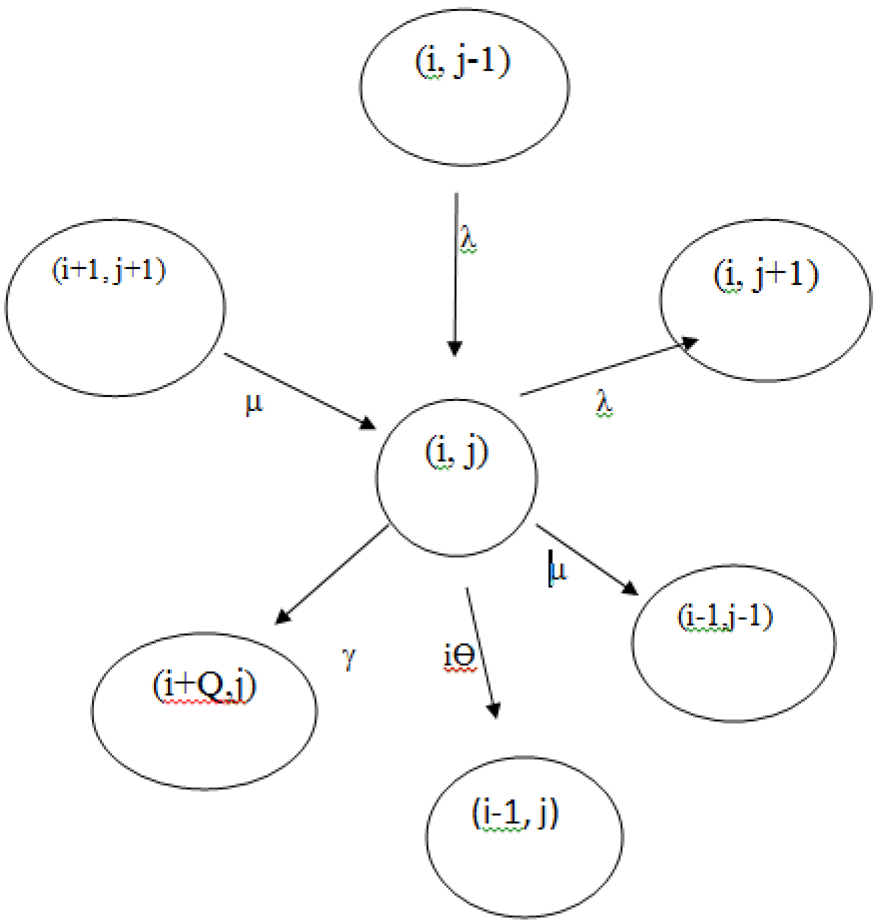
This implies the balance equations (5) – (16) given below. Transition in and out of a states gives the system of equations.

(i) Consider the typical state (j, r) that lies in the range s+1 ≤ j ≤ S-1; 1 ≤ r ≤ N– 1. When (j, r) lies in this range, there is no order pending and hence transition out of this state can be due to either by demand or a service completion. The corresponding balance equation is given by equation (7).

(ii) A service completion in state (j+1, r+1) will decrease both inventory level and number of customers by one unit, thus transition made to state (j, r).

(iii) When one customer arrives and enters the system at state (j, r -1) where r < N, the new state is (j, r). Considering two different ways of reaching state (j, r) both left and right hand side of Eq. (7) are formed.

Hence we have the balance equation (2)-(13) given below. The balance equations can also be obtained by using the fact that transition out of a state is equal to transition into the state. Consider a typical transition rate of a state is equal to transition into a state. For example, let us consider a typical state (j,r) that lies in the range $s+1\le j\le S-1;1\le r\le N-1$. The balance equation of this state is represented in Eq.(12) below. When (j, r) is in this range, there is no order pending, and hence transition out of this state can be only due to either a demand or a service completion. This fact is reflected on the left-hand side of Eq.(12) A service completion in state $(j+1,r+1)$ will decrease both the inventory level and the number of customers by one unit, thus bringing it to state (j,r). State (j,r) can be reached from state $(j+1,r+1)$ when a customer arrives. These are the only two possible ways of reaching state (j,r) and are reflected on the right-hand side of Eq.(12). The state (i,j) will reach to (i-1,j) with state of transition iθ where $\theta > 0$. Fig (2) represent the in-rate and out-rate flow diagram of the general system states (i,j), where $i \in E_1$ and $j \in E_2$.



fig(2)

Now the system of equations can be written in order as follows:

$$(\lambda + S\theta)\pi^f(S,0) = \mu \sum_{j=0}^{S} p_j \pi^f(j,0) \qquad (3)$$

$$(\lambda + \gamma + S\theta)\pi^f(S,r) = \mu \sum_{j=0}^{S} p_j \pi^f(j,r) + \lambda \pi^f(S,r-1), \quad 1 \le r \le N-1 \qquad (4)$$

$$(\gamma + S\theta)\pi^f(\mathrm{S},N) = \mu \sum_{j=0}^{s} p_j \pi^f(j,N) + \lambda\pi^f(\mathrm{S},N-1) \tag{5}$$

$$(\lambda + i\theta)\pi^f(\mathrm{j},0) = \gamma\pi^f(\mathrm{j+1,1}) + (i+1)\theta\pi^f(j+1,0), \quad \mathrm{s}+1 \le j \le S-1 \tag{6}$$

$$(\lambda + \gamma + i\theta)\pi^f(\mathrm{j,r}) = \gamma\,\pi^f(\mathrm{j+1,r+1}) + (i+1)\theta\pi^f(j+1,r) + \lambda\pi^f(\mathrm{j,r-1}),$$
$$\mathrm{s}+1 \le j \le S-1; 1 \le \mathrm{r} \le N-1 \tag{7}$$

$$(\gamma + i\theta)\,\pi^f(\mathrm{j},N) = \lambda\pi^f(\mathrm{j},N-1) + (i+1)\theta\pi^f(j+1,N-1), \quad \mathrm{s}+1 \le j \le S-1 \tag{8}$$

$$(\lambda + \mu p_j + i\theta)\pi^f(\mathrm{j},0) = (i+1)\theta\pi^f(j+1,0) + \gamma\,\pi^f(\mathrm{j+1,1}),\ 1 \le j \le s \tag{9}$$

$$(\lambda + \mu p_j + \gamma + i\theta)\pi^f(\mathrm{j},r) = (i+1)\theta\pi^f(j+1,r) + \gamma\pi^f(\mathrm{j+1},r+1) + \lambda\pi^f(\mathrm{j},r-1), \tag{10}$$

$$1 \le j \le s; 1 \le \mathrm{r} \le N-1,$$

$$(\mu p_j + \gamma + i\theta)\pi^f(\mathrm{j},N) = (j+1)\theta\pi^f(j+1,N) + \lambda\pi^f(\mathrm{j},N-1), 1 \le j \le s \tag{11}$$

$$(\lambda + \mu p_0)\pi^f(0,0) = \gamma\,\pi^f(1,1) + \theta\pi^f(1,0), \tag{12}$$

$$(\lambda + \mu p_0)\pi^f(0,\mathrm{r}) = \gamma\,\pi^f(1,\mathrm{r+1}) + \theta\pi^f(1,r) + \lambda\pi^f(0,\mathrm{r-1}), \quad 1 \le \mathrm{r} \le N-1 \tag{13}$$

$$\mu p_0 \pi^f(0,N) = \lambda\pi^f(0,N-1) + \theta\pi^f(1,N) \tag{14}$$

The above set of equations together with the condition

$\sum_{(j,r)\in E} \Pi^f(j,r) = 1$ determine the steady-state probabilities $\{\pi(i,j)\}$ uniquely. (15)

4.6 System Performance Measures.

(i) The average inventory level in the system is given by

$$\bar{I}^f = \sum_{j=1}^{S} j \sum_{r=0}^{N} \pi^f(\mathrm{j,r}). \tag{16}$$

(ii) Mean waiting time in the system is given by

$$\bar{W}^\pi = \sum_{r=1}^{N} \frac{r}{\gamma} \sum_{j=0}^{s} \pi^f(\mathrm{j,r}) + \sum_{k=0}^{s} \frac{1}{\mu p_k} \sum_{m=1}^{\left[N/s\right]} \sum_{r=1}^{ms} m\pi^f(\mathrm{j,r}). \tag{17}$$

(iii) The mean reorder rate is given by

$$\alpha_a^f = \mu\sum_{r-0}^{N}\sum_{j=0}^{s} p_j\pi^f(\mathrm{j,r}). \qquad (18)$$

(iv) The mean balking rate is given by

$$\alpha_b^f = \lambda\sum_{j=0}^{S}\pi^\pi(\mathrm{j,N}). \qquad (19)$$

(v) The mean service completion rate is given by

$$\alpha_c^f = \gamma\sum_{r=1}^{N}\sum_{j=1}^{S}\pi^\pi(\mathrm{j,r}). \qquad (20)$$

(vi) The expected perishable rate is given by

$$\alpha_d^f = \sum_{j=1}^{S}\sum_{r=0}^{N} i\theta\pi^f(\mathrm{j,r}). \qquad (21)$$

Now the long run expected cost rate is given by

$$C^f = h\sum_{j=0}^{S} j\sum_{r=0}^{N}\pi^f(\mathrm{j,r}) + \frac{c_1}{\gamma}\sum_{r=1}^{N} r\sum_{j=0}^{S}\pi^f(\mathrm{j,r}) + \frac{c_1}{\mu}\sum_{m=1}^{[N/s]}\sum_{r=1}^{ms}\sum_{j=0}^{S}\frac{m}{p_j}\pi^f(\mathrm{j,r}) + c_2\gamma\sum_{r=1}^{N} r\sum_{j=0}^{s} p_j\pi^f(\mathrm{j,r})$$
$$+ g\lambda\sum_{j=0}^{S}\pi^f(\mathrm{j,N}) + \beta\gamma\sum_{j=1}^{S}\sum_{r=1}^{N}\pi^f(\mathrm{j,r}) + \delta\sum_{j=1}^{S}\sum_{r=1}^{N} i\theta\pi^f(\mathrm{j,r}) \qquad (22)$$

4.7 Linear programming problem

4.7.1 LPP Formulation

Let us define the variables D(j, r, k) as follows:

D(j, r ,k)= Pr [decision is k | state is (j,r)]. (23)

Then for any stationary policy f, we have D(j, r, k) =0 or 1.Suppose D(j, r, k) were continuous variable (instead of integers), then the semi-Markov decision problem can be reformulated as a linear programming problem . For this purpose we consider the class of all randomized , time –invariant Markovian policies for which the probability functions D(j, r, k)

satisfy $0 \leq D(j,r,k) \leq 1$ and $\sum_{k \in A_i} D(j.r,k) = 1, 0 \leq r \leq N, 0 \leq j \leq S; i = 1,2$

The linear programming problem is best expressed in terms of the variable

$y(j,r,k)$, which are defined as

$$y(j,r,k) = D(j,r,k)\pi^f(j,r) \tag{24}$$

As y(j, r, k)=Pr[state is (j ,r) and decision is k],for any given f, we have

$$\pi^f(j,r) = \sum_{k \in A} y(j,r,k) \quad (j,r) \in E \tag{25}$$

Expressing $\pi^f(j,r)$ in terms of $y(j,r,k)$ in (21) we obtain the following linear

programming problem:

Minimize

$$C = h\sum_{k \in A} j \sum_{j=1}^{S}\sum_{r=1}^{N} y(j,r,k) + h\sum_{j=1}^{S} j.y(j,0,0) + c_1\mu \sum_{k \in A} \sum_{r=1}^{N} p_j y(j,r,k) + g\lambda y(0,N,0) + c_2 \sum_{k \in A}\sum_{r=1}^{N} \frac{r}{\gamma}\sum_{j=0}^{S} y(j,r,k)$$

$$+\frac{c_2}{\mu}\sum_{k \in A}\sum_{m=1}^{\frac{N}{S}} \sum_{r=1}^{mS}\sum_{j=0}^{S} \frac{m}{p_k} y(j,r,k) + g\lambda \sum_{k \in A}\sum_{j=1}^{S} y(j,N,k) + \sum_{k \in A}\sum_{j=1}^{S} \sum_{r=1}^{N} \beta_k y(j,r,k) + p\sum_{k \in A}\sum_{j=1}^{S}\sum_{r=1}^{N} y(j,r,k) \tag{26}$$

The constraints of the linear programming problem are as follows:

a) From (25), we have

$$y(j,r,k) \geq 0 \quad (j,r) \in E_l, k \in A_l, l = 1,2 \tag{27}$$

b) Since $\sum_{(j,r) \in E} \pi^f(j,r) = 1$ we have from (26)

$$\sum_{l=1}^{2} \sum_{(j,r) \in E_l} \sum_{k \in A_l} y(j,r,k) = 1 \tag{28}$$

As we can see from the lemma below solving the linear programming problem we get an optimal solution when $y\ j,r,k$ are constrained to be integers satisfy (28).

The optimal solution of the above linear programming problem yields a deterministic policy.

From equations (25) and (26), we have

$$D(j, r, k)= \frac{y\ \ j,r,k}{\sum_{k=0}^{K} y\ \ j,r,k}. \tag{29}$$

Since the decision problem is completely ergodic, every basic feasible solution to the above linear programming problem has the property that for each (j,r) ∈ E,D(j,r,k) is 1 for exactly one value of k and zero for all other values of k. Thus ,given the amount of inventory on-hand and the number of customers in the system ,we have to choose the service rate μ_k for which D(j,r,k) is1. Hence any basic feasible solution to the linear programming yields a deterministic policy.

<u>4.7.2 Lemma:</u>

The optimal solution of the above Linear Programming Problem yields a deterministic policy.

<u>Proof:</u>

From the equations

$$y(\mathrm{j,r,k}) = \mathrm{D(j,r,k)}\pi^{f}(\mathrm{j,r}) \tag{30}$$

and

$$\pi^{f}(\mathrm{j,r}) = \sum_{k \in A} y(\mathrm{j,r,k}), (\mathrm{j,r}) \in \mathrm{E}. \tag{31}$$

Since the decision problem is completely ergodic every basic feasible solution to the above linear programming problem has the property that for each

$(\mathrm{j,r}) \in \mathrm{E}, y(\mathrm{j,r,k}) > 0$ for every k∈A.

4.8 Numerical illustration and Discussion

In this section we consider a problem to illustrate the method described in section 4, through numerical examples. We implemented TORA software to solve LPP by simplex algorithm.

We intuitively proposed a conjecture that the reordering rate ($p_j\mu$) to be employed depends only on the inventory level.

This conjecture can be proved for zero lead time and reorder is made at inventory level s and the order quantity is adjusted at the time of replenishment. Sapna, K. P., and Berman, O. [81] proved that the expected cost rate,

$$C(\gamma) = h\left(\frac{S+1}{2}\right) + \sum_{j=0}^{s} \frac{c_1}{s\left[m + \frac{\alpha(1)}{\lambda}\right]} + c_2 \sum_{k\in\{0,1,2\}} \sum_{j=1}^{s} k\ m\ p(\mathrm{j},\mathrm{k})$$

$$+ g\sum_{j=0}^{s} \mathrm{p}(\mathrm{j},\mathrm{N}) + \beta\gamma \sum_{j=1}^{s}\sum_{r=1}^{N} \mathrm{p}(\mathrm{j},\mathrm{r}) + p\sum_{k\in A}\sum_{j=1}^{S}\sum_{r=1}^{N} y(\mathrm{j},\mathrm{r},\mathrm{k})$$

where $m = \frac{1}{\gamma}, \quad \alpha(1) = \frac{1-\frac{\lambda}{\gamma}}{1-\left(\frac{\lambda}{\gamma}\right)^N}$.

For $0 \le \mathrm{j} \le \mathrm{S},\ 1 \le \mathrm{r} \le \mathrm{N}$

$$p\,(\mathrm{j},\mathrm{r}) = \left(\frac{\lambda}{\gamma}\right)^{r-1} P(\mathrm{j},0).$$

For $0 \le \mathrm{j} \le \mathrm{S},\ \mathrm{p}(\mathrm{j},0) = \left(\frac{1}{S}\right)\left(\frac{1-\frac{\lambda}{\gamma}}{1-\left(\frac{\lambda}{\gamma}\right)^N}\right).$

Consider the MDP problem with the following parameters:

S = 3, s = 2, N = 4, λ = 2, μ = 3, γ = 4, h = 0.1, c_j = 3j; j = 0, 1, 2, g = 5, p=2,

$\beta(\mu) = 2\mu$

Action(a)\prob.	p_2	p_1	p_0
0	0.5	0.2	0
1	0.5	0.8	0
2	0.0	0.0	1

The optimum cost for the system is C = 13.96 and Optimal policy is R*(0, 1, 2, 3).

N	4	3	2	1	0
Y_{02}	0.01	0.02	0.03	0.04	0.04
Y_{10}	0.01	0.01	0.03	0.05	0.10
Y_{11}	0	0	0	0	0
Y_{20}	0.01	0.03	0	0.06	0.12
Y_{21}	0	0	0.04	0	0
Y_{30}	0.03	0.04	0.06	0.1	0.18

That is whenever the inventory level reaches the reorder level s(>0), the optimal decision is to refill the inventory with Q=S-s items.

4.6 Conclusions and future research

Analysis of inventory control at service facilities is fairly recent. Most of the earlier work in this direction has been on the determination of ordering policies or on finding optimal stocking levels for a given policy . We approach the problem in a different manner . Given an ordering policy , we determine the service rates to be employed as a function of the number of customers in the queue and the amount of inventory on hand so that the long-run expected cost rate is minimized . The rationale is that quite often it is possible to control the service rate by changing the number of servers or by using a faster or slower server, whereas there may be constraints such as limited storage size and type of vendors that make it difficult to change the stocking level or the frequency of ordering. As such, determination of optimal service rates is an important problem in the service industry.

In our problem, we use an (s, S) ordering policy. Policies such as one-for-one ordering, or (r,Q) r+Q=S systems or any other fixed ordering policy can be analyzed with methodology. We used the tools of Markov decision processes to analyze the problem and linear programming to determine the optimal service rates.

The main contribution of the paper is the determination of the control policy that indicates the specific optimal service rate to be used for every possible state of the system . We made two interesting observations. One is that the service rate to be employed is insensitive to change in inventory level and depend only on the number of customers waiting for service. The second is that the parameters which influence the service rates are the customer arrival rates and waiting time costs . It is not intuitive that the inventory level ,replenishment rate and inventory carrying costs have no effect on the service rate .

Analysis of perishable inventory control system at service facility is fairly recent system study. Most of the previous work determined optimal ordering policies or system performance measures. We approached the problem in a different way, given a service rate we determine the optimal ordering policy to be employed to minimize the long – run expected cost rate. Thus the optimal inventory control with perishable environment in the service facility is established. In future we may extend this model to perishable inventory system with discrete time MDP.

Chapter-5

MDP in Supply Chain: Inventory System with Service Facility with impatient customers

5.1 Introduction

In this chapter, we present a new supply chain model in when MDP is used to take optimized service rate to handle impatient customers. Reneging is phenomena associated with queue when waiting time became intolerable. In most of supply chain, the retailer inventory becomes more critical, because if affects overall efficiency of the chain. In this model consider a SCM system in which retailer inventory is not issued immediately, it delivered to customer after some random service time. Supply Chain exists in both service and manufacturing organizations, but the complexity of the chain may vary greatly from industry to industry.

So, inventory decision is an important component of the supply chain management, because Inventories exist at each and every stage of the supply chain as raw material or semi-finished or finished goods. Since holding of inventories can cost anywhere between 20% to 40% of their value, their efficient management is critical in Supply Chain operations.

The usual objective for a multi-echelon inventory model is to coordinate the inventories and the various echelons so as to minimize the total cost associated with the entire multi-echelon inventory system. This is a natural objective for a fully integrated corporation that operates the entire system. It might also be a suitable objective when certain echelons are managed by either the suppliers or the retailers of the company. The reason is that a key concept of supply chain management is that a company should strive to develop an informal partnership relation with its suppliers and retailers that enable them jointly to maximize their total profit.

Information technology has a substantial impact on supply chains. Scanners collect sales data at the point-of-sale, and Electronic Data Interchange(EDI) allows these data to be shared immediately with all stages of the supply chain. This information system is basis for our model for SCM, and using the MDP tool to get optimal ordering policy at the lowest node (retailer).

Multi-echelon inventory system has been studied by many researchers and its applications in supply chain management has proved worthy in recent literature. Supply chains integrate many operators in the network and optimize the total cost involved without compromising the customer service efficiency

The first quantitative analysis in inventory studies started with the work of Harris, F. [34]. Clark, A.J., and Scarf, H. [18] had put forward the multi-echelon inventory first. They analyzed N-echelon pipelining system without considering a lot size, Recent developments in two-echelon models may be found in Q.M. He and Axsäter, S. [35,10] proposed an approximate model of inventory structure in SC. One of the oldest papers in the field of continuous review multi-echelon inventory system is a basic and seminal paper written by Sherbrooke, C. in 1968 [85]. He assumed (S-1,S) polices in the Deport-Base systems for repairable items in the American Air Force and could approximate the average inventory and stock out level in bases.

Continuous review models of multi-echelon inventory system in 1980's concentrated more on repairable items in a Depot-Base system than as consumable items(see Graves, Moinzadehand, K., and Lee, H. L. [29,55]). All these papers deal with repairable items with batch ordering. Seifbarghy, M., and Jokar, M.R. [83] analyzed a two echelon inventory system with one warehouse and multiple retailers controlled by continuous review (R,Q) policy. A Complete review was provided by Benita M. Beamon (1998) [12]. The supply chain concept grow largely out of two-stage multi-echelon inventory models, and it is important to note that considerable research in this area is based on the classic work of Clark. A.J., and Scarf, H. (1960) [18]. In the case of continuous review perishable inventory models with random lifetimes for the items, most of the models assume instantaneous supply of order. The assumption of positive lead times further increases the complexity of the analysis of these models and hence there are only a limited number of papers dealing with positive lead-times. A continuous review perishable inventory system at Service Facilities was studied by Elango, C. (2001) [21]. A continuous review(s,S) policy with positive lead times in two-echelon Supply Chain was considered by Krishnan, K., and Elango, C. [49].

Bermam, O., and Kim, E. [13] analysed a problem in a stochastic environment where customers arrive at service facilities according to a Poisson Process. The service times are exponentially distributed with mean inter-arrival time assumed to be larger than the order quantity is known. Berman, O., and Sapna, K.P. [14,15] studied an inventory control problem at a service facility requiring one item of the inventory and assumed Poisson arrivals, arbitrarily distributed service times and zero lead times. They assumed finite waiting room. Under a specified cost structure, the optimal ordering quantity that minimizes the long run expected cost rate has been derived.

In this chapter we considered a inventory system maintained in a service facility at Retailer node in tandem supply chain having Retailer vendor(RV) and Distribution Centre(DC). One item from inventory at RV is used to serve the customer and (s, S) ordering policy is adopted at Retail Vendor node. We assume that customers enters the system according to a Poisson arrival process to retail node. The individual customer's unit demand is satisfied after a random time at service which is assumed to have exponential distribution. The lead time of reorders are assumed to have independent exponential distributions. The impatient customers may leave the systems with exponential rate. The joint probability distribution of the number of customers in the waiting room and the inventory level in system is obtained. System performance measures are computed and total expected cost rate is calculated. Optimal reorder quantity is obtained.

5.2 Model Formulation:

- We consider a Supply Chain system consist of Distribution Center(DC), Retail vendor(RV) with service facility and inventory is maintained at both DC and RV nodes with the following operational assumptions.
- For every demand at the retailer node (RV) an item is supplied only after a exponential service time with parameter $\gamma_k : k = 0,1,...,K$.
- The waiting space in the retailer node has maximum capacity N. An arriving customer seeing N customers in the system should leave the system.
- Inventory policy adopted at RV node is (s, S) type in which order for Q=S-s>s items are placed when the inventory level reaches the prefixed level s, and lead time is exponentially distributed with parameter $\mu(>0)$.
- Demand at RV node follows a Poisson process with parameter $\lambda(>0)$.
- At DC, items are packed as Q items in one pocket with maximum inventory level nQ (n pockets). The ordering policy at DC is of (0,nQ) type where the inventory level reach 0, instantaneous replenishment of M=nQ items is made.
- Impatient customers renege from the system with rate α>0(exponentially distributed inter reneging time).
- This system is viewed as a Markov Decision Processes, in which the service rates γ_k (k=0,1,2,3,…, K) are controlled at the service facility in RV node.

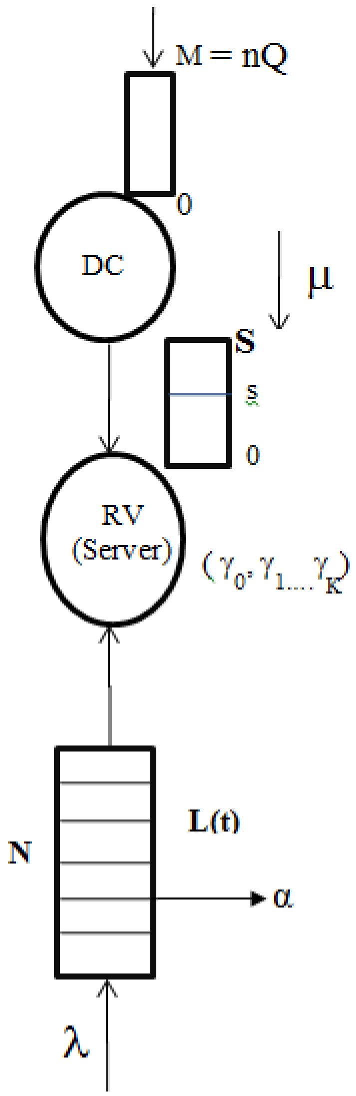


fig (1)Service Facility Systems in Supply Chain with Impatient Customers

Let $I_0(t)$ and L(t) denote the on-hand inventory levels at retailer node and the number of customers in waiting room. Maximum capacity of the waiting room be N. Clearly $\{(I(t),L(t)): t \geq 0\}$ is a Markov Process with state space $E=\{0,1,2,\ldots,S\} \times \{0,1,2,\ldots,N\}$.

We ignore the state of the system (inventory level) at DC, since the replenishment here is instantaneous. Since state space E is finite and all its states are recurrent non-null. $\{(I(t),L(t)): t \geq 0\}$ is an irreducible Markov process with state space E and it is also an Ergodic process. Hence, the limiting distribution exists and is independent of the initial state. In this model our objective is to find an optimal service policy that specifies the expected service rate adopted so as to minimize the long run expected cost rate.

The costs associated with the system operation have the following components:

h: inventory carrying cost per unit time.

c_1: cost per order.

c_2: waiting time cost per unit per unit time.

g: balking cost per customer.

g_1: reneging cost per customer.

β_n : cost associated with using rate γ_n , $\gamma_0 = 0$, $n = 1, 2, ..., \mathrm{K}$.

5.3 Analysis

Let $(I_0^R, \mathrm{L}^R) = \{I_0^R(t); \mathrm{L}^R(t) \geq 0\}$ denote the Markov process , when a stationary policy R is adopted. From our assumptions, it can be seen that the controlled process (I_0^R, L^R) is a finite state semi-Markov decision process. A policy R is called a stationary policy if it is randomized, time invariant and Markovian. Further, a process is said to be completely Ergodic if every stationary policy give rise to an irreducible Markov chain. From our assumptions it can be seen that for every stationary policy f , (I_0^f, L^f) is completely Ergodic. Since the action space is also finite, a stationary optimal policy exists. Hence we consider the class $\Im$ of all stationary policies.

Denote by (k) the action of choosing rate $\gamma_k (k = 0, 1, 2, ..., \mathrm{K})$. Whenever $\mathrm{L}(t) = 0$ or $I_0(t) = 0$, we must choose the rate γ_0. Based on the choice of actions, the state space E can be partitioned as follows:

E_1= $\{(i,q): 0 \leq i \leq S, q = 0; i = 0, 0 \leq q \leq N\}$, E_2= $\{(i,q): 1 \leq i \leq S; 1 \leq q \leq N\}$, E=$E_1 \cup E_2$.

Let A_j(j=1,2) represent the set of all possible actions of the system when it belongs to the set E_j.

Then we have A_1={0}, A_2=$\{(k): 1 \leq k \leq K\}$ and A=$A_1 \cup A_2$

A decision rule from the class ξ is equivalent to a function f:E$\rightarrow$ A and is given by $f(i,q)=\{(k);(i,q)\in E,(k)\in A_j,\ j=1,2\}$.

For any fixed $f\in\xi$ and $(i,q),(j,r)\in E,$ define

$$P_{iq}^{f}(j,r,t)=\Pr\{I_0{}^{f}(t)=j,\mathrm{L}^{f}(t)=r\mid I_0{}^{f}(0)=i,L^{f}(0)=q\},\ (i,q),(j,r)\in E. \tag{1}$$

Then $P_{iq}(j,r,t)$ satisfies the Kolmogorov forward differential equations. As each policy, f, results in an irreducible Markov chain and action spaces are finite,

$P^{f}(j,r)=\lim_{t\to\infty}P_{iq}(j,r,t)$ exists and is independent of the initial state.

Hence we have the balance equation (2)-(13) given below. The balance equations can also be obtained by using the fact that transition out of a state is equal to transition into a state. For example, let us consider a typical state (j,r) that lies in the range $s+1\le j\le S-1;1\le r\le N-1$. This state transition diagram is represented in Eq.(12) below, when (j, r) is in this range, there is no order pending, and hence transition out of this state can be only due to either a demand or a service completion. This fact is reflected on the left-hand side of Eq.(12) A service completion in state $(j,r-1)$ will decrease both the inventory level and the number of customers by one unit, thus bringing it to state (j,r). State (j,r) can be reached from state $(j+1,r+1)$ when a customer arrives. These are the only two possible ways of reaching state (j,r) and are reflected on the right-hand side of Eq.(12). The state (j,r) will reach to (j-1,r) with rate of transition iθ where $\theta\ge 0$.

The balance equations are given by

$$(\lambda+\mu)P^{f}(0,0)=\alpha P^{f}(0,1)+\gamma^{f}P^{f}(1,1) \tag{2}$$

$$(\lambda+\mu+r\alpha)P^{f}(0,r)=\lambda P^{f}(0,r-1)+(r+1)\alpha P^{f}(0,r+1)+\gamma^{f}P^{f}(1,r+1)\quad 1\le r\le N-1. \tag{3}$$

$$(\mu+N\alpha)P^{f}(0,\mathrm{N})=\lambda P^{f}(0,\mathrm{N}-1) \tag{4}$$

For $1 \le j \le s$

$$(\lambda + \mu)P^f(j,0) = \gamma^f P^f(j+1,1). \quad (5)$$

$$(\lambda + \mu + (r-1)\alpha + \gamma^f)P^f(j,r) = \lambda P^f(j,r-1) + r\alpha P^f(j,r+1) + \gamma^f P^f(j+1,r+1), 1 \le r \le N-1. \quad (6)$$

$$(\mu + (N-1)\alpha + \gamma^f)P^f(j,N) = \lambda P^f(j,N-1). \quad (7)$$

For $s+1 \le j \le Q-1$

$$\lambda P^f(j,0) = \gamma^f P^f(j+1,1). \quad (8)$$

$$(\lambda + (r-1)\alpha + \gamma^f)P^f(j,r) = \lambda P^f(j,r-1) + r\alpha P^f(j,r+1) + \gamma^f P^f(j+1,r+1), 1 \le r \le N-1. \quad (9)$$

$$(N-1)\alpha + \gamma^f)P^f(j,N) = \lambda P^f(j,N-1). \quad (10)$$

For $Q \le j \le S-1$,

$$\lambda P^f(j,0) = \gamma^f P^f(j+1,1) + \mu P^f(j-Q,0). \quad (11)$$

$$(\lambda + \gamma^f + (r-1)\alpha)P^f(j,r) = \lambda P^f(j,r-1) + r\alpha P^f(j,r+1) + \gamma^f P^f(j+1,r+1)$$
$$+\mu P^f(j-Q,r), \ 1 \le r \le N-1 \quad (12)$$

$$(\gamma^f + (N-1)\alpha)P^f(j,N) = \lambda P^f(j,N-1) + \mu P^f(j-Q,N) \quad (13)$$

$$\lambda P^f(S,0) = \mu P^f(s,0). \quad (14)$$

$$(\lambda + (r-1)\alpha + \gamma^f)P^f(S,r) = \lambda P^f(S,r-1) + r\alpha P^f(S,r+1) + \mu P^f(s,r), \quad 1 \le r \le N-1. \quad (15)$$

$$((N-1)\alpha + \gamma^f)P^f(S,N) = \lambda P^f(S,N-1) + \mu P^f(s,N). \quad (16)$$

The above set of equations together with the condition

$$\sum_{(j,r) \in E} P^f(j,r) = 1 \quad (17)$$

determine the steady-state probabilities uniquely.

5.4 System Performance Measures

The steady state probability $P^f(j, r)$ also gives the long-run fraction of time the system is in the state (j, r),

(i) The mean inventory level in the system is given by

$$\bar{I}^{f} = \sum_{j=1}^{s} j \sum_{r=0}^{N} P^{f}(j,r).$$

(18)

2. The expected cost due to different service rates utilized is

$$\bar{\Gamma}_{I^f} = \sum_{(j,r)\in E} \Gamma_{f(r)} P^{f}(j,r), \text{where } \Gamma_{f(r)} = \beta_n \ if \ f(r) = n \tag{19}$$

3. The mean reorder rate is given by

$$\bar{\alpha}_1{}^{f} = \gamma^{f} \sum_{r=0}^{N} P^{f}(s+1,r). \tag{20}$$

4. The mean waiting time in the system is given by

$$\bar{\alpha}_2{}^{f} = \sum_{r=1}^{N} \frac{r}{\gamma^{f}} \sum_{j=0}^{S} P^{f}(j,r) + \frac{1}{\mu} \sum_{m=1}^{[N/S]} \sum_{r=1}^{mS} m \sum_{j=0}^{S} P^{f}(j,r). \tag{21}$$

5. The mean balking rate is given by

$$\bar{\alpha}_3{}^{f} = \lambda \sum_{j=0}^{S} P^{f}(j,N). \tag{22}$$

6. The mean reneging rate is given by

$$\bar{\alpha}_4{}^{f} = \sum_{j=0} \sum_{r=1}^{S} j\alpha P^{f}(j,\mathrm{r}) \tag{23}$$

The long-run expected cost rate when policy f is adopted is given by

$$C^{f} = h\bar{I}^{f} + c_1\alpha_1^{f} + c_2\alpha_2^{f} + g\alpha_3^{f} + \bar{\Gamma}_{I^f}, \tag{24}$$

where in the steady state, for a given policy f ,

The following cost structure is imposed on the system. $\bar{I}^{f}$ is the average inventory level, α_1^{f} is the expected reorder rate, α_2^{f} is the average waiting time for a customer; α_3^{f} is the expected balking rate, and $\bar{\Gamma}_{I^f}$ is the expected cost per unit time associated with using the different rates.

Our objective is to find an optimal policy f^* for which $C^{f^*} \le C^{f}$ for every f .

Hence the average cost rate of the system is given by

$$C^f = h\sum_{j=1}^{S} j\sum_{r=0}^{N} P^f(j,r) + c_1\gamma^f \sum_{r=0}^{N} P^f(s+1,r) + c_2\sum_{r=1}^{N}\frac{r}{\gamma_r{}^f}\sum_{j=0}^{S} P^f(j,r)$$

$$+\frac{c_2}{\mu}\sum_{m=1}^{[N/S]}\sum_{r=1}^{mS} m\sum_{j=0}^{S} P^f(j,r) + g\lambda\sum_{j=0}^{S} P^f(j,N) + \sum_{(j,r)\in E}\Gamma_{f(r)}P^f(j,r) + g_1\sum_{j=0}^{S}\sum_{r=1}^{S} j\alpha P^f(j,\mathrm{r}).$$

(25)

5.5 Linear programming problem

5.5.1 LPP Formulation

Let us define the variables D(j, r, k) as follows:

D(j, r ,k)= Pr [decision is k | state is (j,r)]. Then for any stationary policy f, we have D(j, r, k) =0 or 1.Suppose D(j, r, k) were continuous variable (instead of integers), then the semi-Markov decision problem can be reformulated as a linear programming problem . For this purpose we consider the class of all randomized , time –invariant Markovian policies for which the probability functions D(j, r, k) satisfy

$$0 \leq D\ \ j,r,k\ \ \leq 1, \qquad k = 0,1,\ldots,K$$

and

$$\sum_{k\in A_i} D(j,r,k) = 1, 0 \leq r \leq N, 0 \leq j \leq S; i = 1,2,\ k = 0,1,\ldots,K.$$

The linear programming problem is best expressed in terms of the variables $y(j,r,k)$, which are defined as follows:

$$y(j,r,k) = D(j,r,k)P^f(j,r) \qquad (26)$$

As y(j, r, k)=Pr[state is (j ,r) and decision is k],for any given f, we have

$$P^f(j,r) = \sum_{k\in A} y(j,r,k)\ (\mathrm{j,r})\in \mathrm{E}\ ,\ k = 0,1,\ldots,K. \qquad (27)$$

Expressing $P^f(j,r)$ in terms of $y(j,r,k)$ in (22) we obtain the following linear programming problem:

$$Min\ C = h\sum_{k=1}^{K} j\sum_{j=1}^{S}\sum_{r=1}^{N} y(j,r,k) + h\sum_{j=1}^{S} jy(j,0,0) + c_1\sum_{k=1}^{K}\sum_{r=1}^{N}\gamma_k y(s+1,r,k) + g\lambda y(0,N,0) + c_2\sum_{k=1}^{K}\sum_{r=1}^{N}\frac{r}{\mu_k}\sum_{j=0}^{S} y(j,r,k)$$

$$+c_2\sum_{k=1}^{K}\frac{1}{\mu}\sum_{m=1}^{\frac{N}{S}}\sum_{r=1}^{mS} m\sum_{j=0}^{S} y(j,r,k) + g\lambda\sum_{k=1}^{K}\sum_{j=1}^{S} y(j,N,k) + \sum_{(j,r)\in E, r\neq 0}\sum_{k=1}^{K}\mu_k y(j,r,k) + g_1\sum_{j=0}^{s}\sum_{k=1}^{K} j\alpha y(j,\mathrm{r},\mathrm{k}) \quad (28)$$

The constraints of the linear programming problem are expressed as follows:

a) From (26), we have

$$y(j,r,k) \geq 0 \ (j,r) \in E_l, k \in A_l, l = 1,2 \quad (29)$$

b) Since $\sum_{(j,r)\in E} P^f(j,r) = 1$ we have from (27)

$$\sum_{l=1}^{2}\sum_{(j,r)\in E_l}\sum_{k\in A_l} y(j,r,k) = 1 \quad (30)$$

c) The remaining constraints are the balance equations (2) to (16).

As we can see from the lemma below solving the linear programming problem gives the optimal solution when the $y\ j,r,k\ s$ are constrained to be integers.

The optimal solution of the above linear programming problem yields a deterministic policy.

From equations (25) and (26), we have

$$\mathrm{D(j,\ r,\ k)} = \frac{y\ \ j,r,k}{\sum_{k=0}^{K} y\ \ j,r,k} \quad (31)$$

Since the decision problem is completely ergodic, every basic feasible solution to the above linear programming problem has the property that for each (j,r) ∈ E,D(j,r,k) is 1 for exactly one value of k and zero for all other values of k. Thus given the amount of inventory on-hand and the number of customers in the system ,we have to choose the service rate γ_k for which

D(j,r,k) is1. Hence any basic feasible solution of the linear programming yields a deterministic policy.

5.6 Numerical illustration and discussion

In this section ,we illustrate the method described in section 4 through numerical examples . We use the simplex package in TORA for solving the linear programming problem . In all cases the computational time was less than 10 s on a PENTIUM 3.0 alpha machine.

It is our conjecture that the service rate to be employed depends only on the number of customers in the system and not on the inventory level. This conjecture can be proved for the zero-leadtime case assuming that the same γ (service rate) is used for all states. Then the expected cost rate is given by,

$$C(\mu)=\frac{c_1}{S[m+\alpha(1)/\lambda]}+h\frac{S+1}{2}+c_2\sum_{k=1}^{K}\sum_{j=1}^{S}kmP(j,k)+g\sum_{j=0}^{S}P(j,N)+\mu(\gamma)\sum_{j=1}^{S}\sum_{r=1}^{N}P(j,r), \\ +g_1\sum_{j=0}^{s}\sum_{k=1}^{K}j\alpha y(j,\mathrm{r},\mathrm{k}) \tag{32}$$

where m =1/ μ and

$$\alpha(1)=\frac{1-(\lambda/\gamma)}{1-(\lambda/\gamma)^N} \tag{33}$$

For $0\le j\le S;\ 1\le r\le N,$

$$P(j,r)=\left[\frac{\lambda}{\gamma}\right]^{r-1}P(j,0) \tag{34}$$

For $0\le j\le S,$

$$P(j,0)=\frac{1}{S}\frac{1-(\lambda/\gamma)}{1-(\lambda/\gamma)^{N+1}} \tag{35}$$

In the above equations, $\mu(\gamma)$, an increasing function of γ, is the cost associated with service rate γ . By differentiating C(γ) with respect to γ, it is easy to verify that the stationary value of γ is independent of the inventory level. When different service rates are used for different states we were unable to prove the conjecture. However, from the numerous numerical examples we ran, we found that the service rate is insensitive to changes in the inventory level and replenishment rates. Furthermore, service rates also seem to be independent of ordering costs, inventory carrying costs and balking costs.

The only three parameters that have appreciable effect on the service rates are the arrival rate, the waiting time costs and the number of customers in the system.

Typical numerical results, from one of the examples we studied, are summarized in Tables 1-2, where n is the number of customers in the system. The first two tables show the effect of changes in waiting time costs .In the above two tables, we use γ_1=2, γ_2=3, γ_3=4, γ_4=5.

Two immediate conclusions can be drawn from the computational results in Tables 1-2:

(1) As the cost of waiting, c2, increases, larger service rates are utilized for larger values of n. For examples, in Table 1,when c_2=1, γ_1=2 is used for all n values, expect for n=9,10, where γ_2=3 is utilized .When c_2=10 , γ_1=2is not used at all and γ_3, γ_4, γ_5 are used for respectively $1 \le n \le 3$, $4 \le n \le 5$ and $6 \le n \le 10$.

(2) As the arrival rate λ is increasing , larger service rates are utilized for larger values of n . We note that for a service facility, a large number of customers (n) waiting is equivalent to poor service. Therefore the conclusions above are quite important as they show that the model can be used to decrease the average waiting time (providing good service) by selecting the appropriate service rates in a cost effective way.

Tables 1

Optimal rates for c_1=50,h=0.1,g=2,

$\mu_1 = 1, \mu_2 = 2, \mu_3 = 3, \mu_4 = 4, S = 25, s = 6, N = 10, \lambda = 7, \gamma = 0.3$

C_2	γ_1=2	γ_2=3	γ_3=4	γ_4=5
1	$1 \le n \le 9$	$10 \le n \le 12$	_	_
2	$1 \le n \le 6$	$7 \le n \le 10$	_	_

3	$1 \le n \le 5$	$6 \le n \le 9$	_	_
4	$1 \le n \le 4$	$5 \le n \le 8$	n=9	_
5	$1 \le n \le 3$	$4 \le n \le 7$	$7 \le n \le 9$	_
6	$1 \le n \le 2$	$3 \le n \le 7$	$8 \le n \le 9$	n=10
7	$1 \le n \le 2$	$3 \le n \le 6$	$7 \le n \le 8$	$9 \le n \le 10$
8	$1 \le n \le 2$	$3 \le n \le 5$	$6 \le n \le 8$	$9 \le n \le 10$
9	$n = 1$	$2 \le n \le 3$	$4 \le n \le 6$	$7 \le n \le 10$
10	_	$1 \le n \le 3$	$4 \le n \le 5$	$6 \le n \le 10$

Tables 2

Optimal rates for c_1=50,h=0.1,g=2,

$\mu_1 = 1, \mu_2 = 2, \mu_3 = 3, \mu_4 = 4, S = 25, s = 6, N = 10, \lambda = 7, \gamma = 0.3$

C_2	γ_1=2	γ_2=3	γ_3=4	γ_4=5
1	$1 \le n \le 9$	n=10	_	_
2	$1 \le n \le 8$	$9 \le n \le 10$	_	_
3	$1 \le n \le 7$	$8 \le n \le 9$	n=10	_
4	$1 \le n \le 6$	$7 \le n \le 8$	$9 \le n \le 10$	_
5	$1 \le n \le 4$	$5 \le n \le 6$	$7 \le n \le 10$	_
6	$1 \le n \le 4$	$5 \le n \le 6$	$7 \le n \le 9$	n=10
7	$1 \le n \le 4$	n=5	$6 \le n \le 8$	$9 \le n \le 10$
8	$1 \le n \le 3$	n=4	$5 \le n \le 7$	$8 \le n \le 10$
9	$1 \le n \le 2$	_	$3 \le n \le 7$	$8 \le n \le 10$
10	n=1	n=2	$3 \le n \le 6$	$7 \le n \le 10$

In Table1, the costs associated with the different rates are linear functions of the rates ($\gamma_1 = 2\gamma$ and $\gamma_2 = 3\gamma$,respectively). In Table 2 where there is a steeper increase in a cost for changing the rate from γ_1 to γ_2 ($\gamma_1 = 2\gamma = 1,\ \gamma_2 = 3$), the number of states for which rate is γ_2 is smaller than that in Table 1. For example when c_2=9, there is no state in which rate γ_2 is employed and when c_2=1, 7,8,10, there is only one state in which rate γ_2 is employed.

The case where $\gamma_n = n\gamma$ (,n is an integer) can be interpreted as a system in which n servers are used each serving at a rate γ. Hence, the problem of determining the number of servers to be employed as a function of the state of the system can be solved by using our model by simply taking $\gamma_n = n\gamma$

5.7 Conclusions and future research

Analysis of inventory control at service facilities is fairly recent. Most of the earlier work in this direction has been on the determination of ordering policies or on finding optimal stocking levels for a given policy. We approach the problem in a different manner . Given an ordering policy, we determine the service rates to be employed as a function of the number of customers in the queue and the amount of inventory on hand so that the long-run expected cost rate is minimized. The rationale is that quite often it is possible to control the service rate by changing the number of servers or by using a faster or slower servers, whereas there may be constraints such as limited storage size and type of vendors that make it difficult to change the stocking level or the frequency of ordering. As such, determination of optimal service rates is an important problem in the service industry.

In our problem, we use an (s, S) ordering policy and Markov decision processes tool to analyze the problem via linear programming to determine the optimal service rates.

The main contribution of the paper is the determination of the control policy that indicates the specific optimal service rate to be used for every possible state of the system. We made two interesting observations.

One is that the joint probability distribution of the number of customers in the waiting room and the inventory level in the system is obtained. System performance measures are computed and total expected cost rate is calculated. The second is that the parameters which influence the service rates are the customer arrival rates and waiting time costs of the inventory model.

Chapter-6

Optimal Inventory Control with Partial Backlogging in a Supply Chain: MDP Approach

6.1 Introduction

In this chapter we present a MDP based Supply Chain Inventory management with partial backlogging for highly demanding branded items, supply by the network may not be sufficient in the case at least partial backlogging of demand is essential. In this model we considers a SCM system in which MDP is applied to take optimal ordering decision so that the backlogging solved be cleared within a specified time period. For this case minimization of total expected cost rate is taken as a criterion.

The study of Supply Chain Management(SCM) started in the late 1980s and has gained a growing level of interest from both companies and researchers over the past three decades. There are many definitions of Supply Chain management. A supply Chain may be defined as an integrated process wherein a number of various business entities (i.e suppliers, manufacturers, distributors and retailers) work together in an effort to (i)acquire raw materials (ii) Convert these raw materials into specified final products and (iii) deliver these final products to retailers. The process and delivery of goods through this network needs efficient communication and transportation system. The supply chain is traditionally characterized by a forward flow of materials and products and backward flow of information.
One of the most important aspects of supply chain management is inventory control. Inventory control models are almost invariably stochastic optimization problems with objective function being either expected costs or expected profits or risks. In practice, a retailer may want an optimal decision which achieves a minimal expected cost or a maximal expected profit with low risk of deviating from the objective.

A complete review of SCM was provided by Benita M. Beamon (1998) [12]. However, there has been increasing attention placed on performance, design and analysis of the supply chain as a whole. HP's (Hawlett Packard) Strategic Planning andModelling (SPM) group initiated this kind of research in 1977. With-in manufacturing research, the supply chain concept grow largely out of two-stage multi-echelon inventory models, and it is important to note that considerable research in this area is based on the classic works of Clark, A.J., and Scarf, H. (1960) [18] and Sherbrooke,C. [85]

Recent developments in two-echelon models may be found in He, Q.M., and Jewkes, E.M. (2000) [35], Axaster, S., (1993) [9], Nahimas, S. (1980) [61].

This chapter deals with a simple MDP, supply chain model that is modelled as system with a single warehouse, a distribution centre and single retailer (all retailers are identical in character), handling a single product. In order to avoid the complexity, at the same time without loss of generality, we assumed the Poisson demand pattern at retailer node. This restricts our study to design and analyse as the tandem network of inventory, which is the building block for the whole supply chain system. Also we assume partial backlogging due to stock situation.

The rest of the chapter is organized as follows; the model formulation is described in section 2. In section 3, steady state analysis and MDP formulation are done. Section 4deals with the LPP solution procedure for the system in steady state Numerical example and sensitivity analysis are provided in section 5. The last section 6 concludes the chapter.

6.2 The Model Description

We consider a continuous review two-echelon inventory system in supply chain implementing partial backlogging at retailer node. A finished product is supplied from distribution centre to retailer who adopts (s,S) policy for maintaining inventory. The demand at retailer node follows a Poisson process with mean rate $\lambda>0$. Supply to the retailer in pockets of $Q=S-s,>s$ items is administered with exponential lead time having parameter $\mu_0>0$. The replenishment of items in terms of pockets at distribution centre is made from a Ware house (WH) with one for one policy (n-1,n). Demands occurred during the stock out periods are backlogged up to a specified quantity 'b' at retailer node(where b is the backorder level such that Q>b+s (that is Q-s>b>0). In this model, the maximum inventory levels M and S are fixed and reorder level is assumed to be s such that S-s=Q>b+s and M=nQ, $n \in N$. The refilling of warehouse is instantaneous from the manufacture.

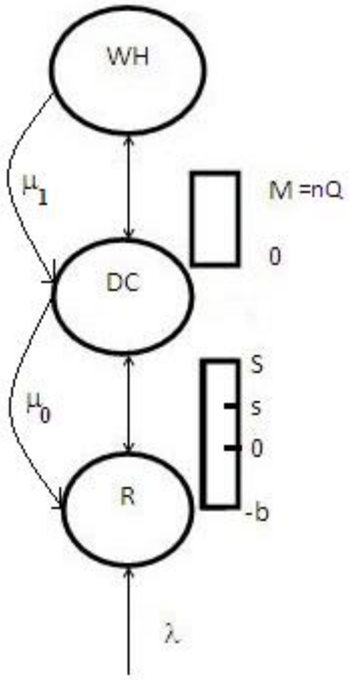


Figure1: Inventory Control with Partial Backlogging in Supply Chain

6.3 Analysis

6.3.1 System Analysis

Let $I_0(t)$ and $I_1(t)$ denote the on hand inventory level at retailer node and distribution centre respectively at time t. Then $I(t) = \{I_0(t), I_1(t) : t \geq 0\}$ is a Markov process with state space $E = E_1 \times E_2$ where $E_1 = \{S, (S-1), \ldots, s, (s-1), \ldots, 2, 1, 0, -1, -2, -3, \ldots, -b\}$, $E_2 = \{nQ, (n-1)Q, \ldots, Q, 0\}$, where b is partial backorder level such that Q > b+s that is$0 \leq b \leq Q-s$(we use negative sign for backlogging quantity.

State transitions for some specific states are given below:

(i) The arrival of a demand for an item at retailer node makes a state transition in the Markov Process from (j,q) to (j-1,q) with rate λ.

(ii) Replenishment of Q items at retailer node makes a state transition from (j,nQ) to (j+Q, (n-1)Q) with rate $\mu_0>0$.

(iii) Replenishment of inventory at distribution centre node makes a state transition from (j, (n-1)Q) to (j, nQ) with rate $\mu_1>0$.

6.3.2 MDP Formulation:

Decision epochs: Random time points at when demand occurs are taken as decision epochs, since the system state changes at that time.

State Space: The state space E is given by$E=E_1\times E_2$,

where $E_1= \; S,(S-1),...,s,(s-1),...,2,1,0,-1,-2,-3,...,-b \;$, $E_2 = \; nQ,(n-1)Q,...,Q,0 \;$,

where b is partial backorder level such that Q>b+s that is$0\leq b\leq Q-s$(we use negative sign for backlogging quantity).

Action Set: The reordering decisions (0- no order;1- order; 2 –compulsory order) taken at each state of the system (j, q) $\in$ E. The compulsory order for S items is made when inventory level is zero. Let A_i (i =1, 2, 3) denotes the set of possible actions. $A_1 = \{0\}$, $A_2 = \{0, 1\}$, $A_3 = \{2\}$ and $A = A_1\cup A_2\cup A_3$.

The set of all possible actions are at (i,j)$\in$ E is given by

$$A_{(i,j)} = \begin{cases} \{0\}, & s+1\leq i\leq S, \quad 1\leq j\leq n \\ \{0,1\}, & 1\leq i\leq s, \quad\quad\quad 1\leq j\leq n \\ \{2\}, & i=0, \quad\quad\quad\quad 1\leq j\leq n \end{cases}$$

The state space E can be partition in to

$$E_1 = \; (i,j)/s+1\leq i\leq S, \quad 1\leq j\leq n$$

$$E_2 = \; (i,j)/1\leq i\leq s, 1\leq j\leq n$$

$$E_3 = \; (0,j)/1\leq j\leq n$$

Transition Probability: $p_{(j,q)}^{(k,r)}(a)$ denote the transition probability from state (j, q) to state (k, r) when decision a is made at state (j, q). The transition probabilities can be obtained from the infinitesimal generator R of the Markov Process $\{I_0(t), I_1(t) : \text{t} \geq 0\}$

$$R = \begin{matrix} nQ \\ (\text{n}-1)\text{Q} \\ (\text{n}-2)\text{Q} \\ (\text{n}-3)\text{Q} \\ \vdots \\ Q \\ 0 \end{matrix} \begin{pmatrix} A & B & 0 & 0 & \cdots & 0 & 0 \\ C & D & B & 0 & \cdots & 0 & 0 \\ 0 & C & D & B & \cdots & 0 & 0 \\ 0 & 0 & C & D & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & D & B \\ 0 & 0 & 0 & 0 & \cdots & C & E \end{pmatrix}$$

Cost function:

$c_{(j,q)}(a)$ denote the cost occurred in the system when action'a' is taken at state (j, q).

6.4 System Performance Measures

1. **Mean Reorder Rates:**

(a) Consider the event β_i of reorders at node i (i=0, 1). Observe that β_0 denote the event that the inventory level at retailer node reaches s, whereas the β_1 an event that the inventory level at distribution centre reaches the level M-Q (i.e. n-1 packets). The mean reorder rate at retailer node is given by

$$\beta_0 = \lambda \sum_{q=0}^{nQ} {}^{*}v_{s+1}^{q}$$

(b) The mean reorder rate at the distribution centre is given by

$$\beta_0 = \mu_0 \sum_{j=0}^{s} \sum_{q=0}^{(\text{n}-1)Q} {}^{*}v_{j}^{q}$$

2. **Mean Inventory Levels:**

Let $\overline{I_i}$ denote the mean inventory level in the steady state at node i (i=0, 1).

Thus the mean inventory level at retailer node is given by

$$\overline{I}_0 = \sum_{q=0}^{nQ}{}^{*}\left(\sum_{j=0}^{S} j \cdot v_j^q\right)$$

The mean inventory level at distribution centre is given by

$$\overline{I}_1 = \sum_{j=0}^{S}\left(\sum_{q=0}^{nQ}{}^{*} q \cdot v_j^q\right).$$

3. Mean Shortage Rates:

Let α_0 denote the shortage rate at retailer node. Thus the shortage rate at retailer node is given by $\alpha_0 = \lambda \sum_{q=0}^{nQ}{}^{*} v_{-b}^q$

The shortage rate at distribution centre α_1 is given by

$$\alpha_1 = \mu_0 \sum_{q=0}^{nQ} {}^{*} v_j^0.$$

Cost Analysis:

The long-run expected cost rate when policy f is adopted is given by

$$C(s,Q) = h_0 \overline{I}_0 + h_1 \overline{I}_1 + k_0\beta_0 + k_1\beta_1 + g_0\alpha_0$$

where k_0 and k_1 are setup costs (ordering cost) regardless of order size; h_0 and h_1 be holding cost per unit item per unit time at retailer and distributor nodes respectively; g_i is the shortage cost at node i (i=0, 1) for unit shortage per unit time.

Hence the average cost rate of the system is given by

$$\begin{aligned} C(s,Q) = {} & h_0\left(\sum_{q=0}^{nQ}{}^{*}\left(\sum_{j=0}^{S} j \cdot v_j^q\right)\right) + h_1\left(\sum_{j=0}^{S}\left(\sum_{q=0}^{nQ}{}^{*} q \cdot v_j^q\right)\right) + k_0\left(\lambda\sum_{q=0}^{nQ}{}^{*} v_{s+1}^q\right) \\ & + k_1\left(\mu_0 \sum_{j=0}^{s}\sum_{q=0}^{(n-1)Q}{}^{*} v_j^q\right) + g_0\left(\lambda \sum_{q=0}^{nQ}{}^{*} v_{-b}^q\right) \end{aligned}$$

When policy R is adopted. From our assumptions, it can be seen that the controlled process (I_0^R, I_1^R) is a finite state semi-Markov decision process. A policy R is called a stationary policy if it is randomized, time invariant and Markovian. Further, a process is said to be completely ergodic if every stationary policy give rise to an irreducible Markov chain. From our assumptions it can be seen that for every stationary policy f, (I_0^f, I_1^f) is completely ergodic. Since the action space is also finite, a stationary optimal policy exists. Hence we consider the class f of all stationary policies.

Our objective is to find an optimal policy $f*$ for which $C^{f*} \leq C^{f}$ for every f.

For any fixed $f \in F$ and $(\text{i, j}),(\text{k,l}) \in E$, define

$$P_{jq}^{f}(\text{k,l},t) = \Pr[I_0{}^{f}(t) = k, I_1{}^{f}(t) = l \mid I_0{}^{f}(0) = i, I_1{}^{f}(0) = j], \ (\text{i, j}),(\text{k,l}) \in E$$

Then $P_{jq}(\text{k},l,t)$ satisfies the Kolmogorov forward differential equations. As each policy, f, results in an irreducible Markov chain and action spaces are finite,

$P^{f}(\text{k,l}) = \lim P_{jq}(\text{k,l},t)$ exists and is independent of the initial conditions.

The condition

$\sum_{(\text{j,k}) \in E} P^{f}(\text{i, j}) = 1$, determine the steady-state probabilities uniquely.

6.5 Linear programming problem

6.5.1 LPP Formulation

Let us define the variables D(i, j, a) as

D(i, j, a)= Pr [decision is a \ state is (i, j)].

Then for any stationary policy f, we have D(i, j, a) =0 or 1.Suppose D(i, j, a) were continuous variable (instead of integers), then the semi-Markov decision problem can be reformulated as a linear programming problem . For this purpose we consider the class of all randomized , time –invariant Markovian policies for which the probability functions D(j, r, k) satisfy

$$0 \leq D(i,j,a) \leq 1$$

and

$$\sum_{a \epsilon A} D(i,j,a) = 1, 0 \leq i \leq S, 0 \leq j \leq M$$

The linear programming problem is best expressed in terms of the variables y(i, j, a), which are defined as y(i,j, a) =D(i, j, a)P^f(i,j), (1)

As y(i, j, a)=Pr[state is (i ,j) and decision is a],for any given policy f.

$p^f(i,j)= \sum_{a\epsilon A} y\ (i,j,a)\ \ (i,j)\epsilon E$ (2)

Expressing P^f(i, j) in terms of y(i, j, a) , we obtain the following linear programming problem:

Minimize

$$C(s,Q) = h_0\left(\sum_{a\in\{0,1,2\}}\sum_{j=0}^{nQ}{}^{*}\left(\sum_{i=0}^{S} i\cdot v_i^j(a)\right)\right) + h_1\left(\sum_{a\in\{0,1,2\}}\sum_{i=0}^{S}\left(\sum_{j=0}^{nQ}{}^{*}\ j\cdot v_i^j(a)\right)\right) + k_0\left(\sum_{a\in\{0,1,2\}}\lambda\sum_{j=0}^{nQ}{}^{*} v_{s+1}^j(a)\right)$$

$$+ k_1\left(\sum_{a\in\{0,1,2\}}\mu_0\sum_{i=0}^{s}\sum_{j=0}^{(n-1)Q}{}^{*} v_i^j(a)\right) + g_0\left(\sum_{a\in\{0,1,2\}}\lambda\sum_{j=0}^{nQ}{}^{*} v_{-b}^j(a)\right) + g_1\left(\sum_{a\in\{0,1,2\}}\mu_0\sum_{j=0}^{nQ} v_i^0(a)\right)$$

The constraints of the linear programming problem are as follows:

a)$y\ (i,j,a)\ \geq 0$ (i,j)∈ $E, a\in A_k$, k=0,1,2. (3)

b) $\sum_{(i,j)\in E} P^f(i,j) = 1$ and

$$\sum_{i=0}^{2}\sum_{(i,j)\in E_l}\sum_{k\in A_l} y(i,j,k) = 1$$

c) $$\sum_{i=0}^{2}\sum_{(i,j)\in E}\sum_{a\in A_i} y(i,j,a) = 1,$$ (4)

d) The remaining constraints are the balance equations

As we can see from the lemma below solving the linear programming problem gives the optimal solution when the $y\ (i,j,k)$ are constrained to be integers.

The optimal solution of the above linear programming problem yields a deterministic policy.

From equations (1) and (2), we have

$$D(j,r,k)=\frac{y\ (i,M,k)}{\sum_{k=0}^{K} y\ (i,j,k)} \quad (5)$$

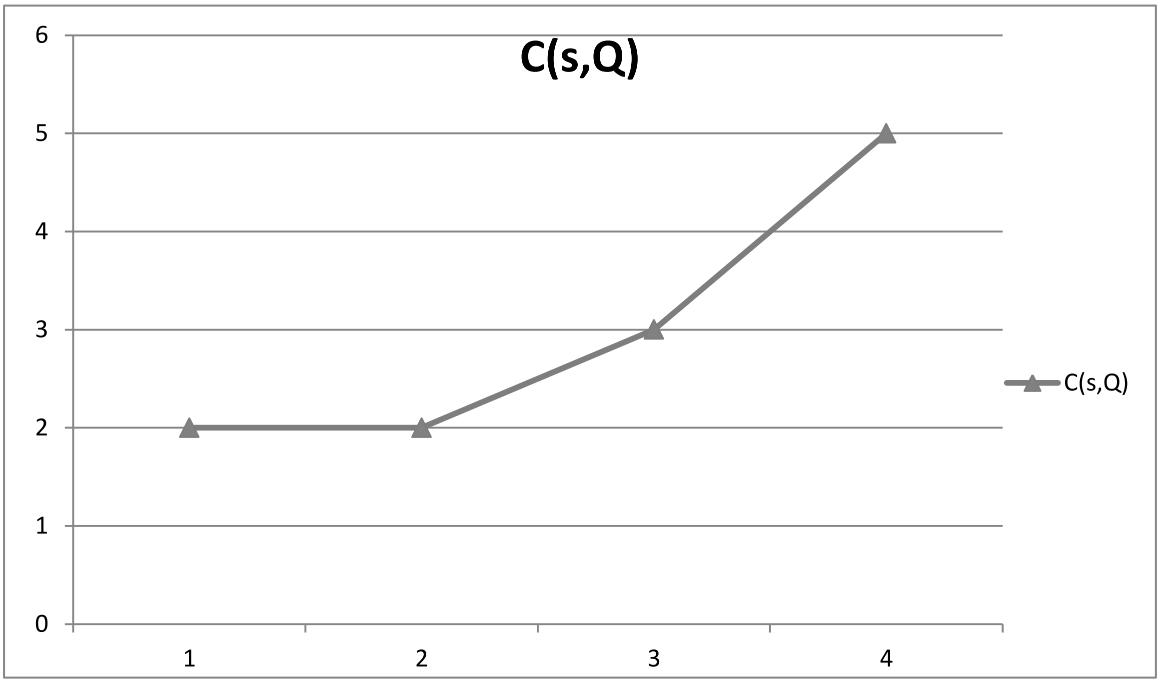
Since the decision problem is completely ergodic, every basic feasible solution to the above linear programming problem has the property that for each(i,j)∈ E,D(i,j,k) is 1 for exactly one value of k and zero for all other values of k. Thus ,given the amount of inventory on-hand and the number of customers in the system ,we have to choose the service rate γ_k for which D(i,j,k) is1.Hence any basic feasible solution of the linear programming yields a deterministic policy.

6.6 Numerical illustration and discussion:

Total excepted cost rate C(s, Q) as a function s and Q is given in the following table

s	**Q**	**C(s,Q)**
0	7	8.0002
1	6	7.8571
2	5	7.4543
*3	*4	*6.4932
4	3	7.6702
5	2	7.7945
6	1	7.9820

The graphical representation of the long run expected cost rate C(s*,Q*) is given below:



6.7 Conclusion:

Analysis of inventory control in Supply Chain has been studied by many researchers in the recent past. But most of them are in the direction of determining, the optimal inventory level (MDP), when a fixed policy is implemented. We, in this model studied the problem in a new approach. For a given Supply Chain structure(two-stage), the ordering quantities are controlled with MDP, and the optimal ordering policy is founded. For both DC and Retailer only partial backlogging is considered. One may extend this full backlogging in future.

Chapter-7

Optimal Control for Perishable Inventory with Partial Backlogging in a Supply Chain: MDP Approach

7.1 Introduction

The study of Supply Chain Management (SCM) started in the late 1980s and has gained a growing level of interest from both companies and researchers over the past three decades. There are many definitions of Supply Chain management. A supply Chain may be defined as an integrated process wherein a number of various business entities(i.e suppliers, manufacturers, distributors and retailers) work together in an effort to (i)acquire raw materials (ii) Convert these raw materials into specified final products and (iii) deliver these final products to retailers. The process and delivery of goods through this network needs efficient communication and transportation system. The supply chain is traditionally characterized by a forward flow of materials and products and backward flow of information. One of the most important aspects of supply chain management is inventory control. Inventory control models are almost invariably stochastic optimization problems with objective function being either expected costs or expected profits or risks. In practice, a retailer may want an optimal decision which achieves a minimal expected cost or a maximal expected profit with low risk of deviating from the objective.

A complete review of SCM was provided by Benita, M., Beamon (1998) [12]. However, there has been increasing attention placed on performance, design and analysis of the supply chain as a whole. HP's (Hawlett Packard) Strategic Planning and Modelling (SPM) group initiated this kind of research in 1977. With-in manufacturing research, the supply chain concept grow largely out of two-stage multi-echelon inventory models, and it is important to note that considerable research in this area is based on the classic works of Clark, A.J., and Scarf, H. (1960) 18] and Sherbrooke, C. [85].

Recent developments in two-echelon models may be found in . He, Q.M., and Jewkes, E.M. (2000) [35], Axaster, S. (1993) [9], Nahimas, S. (1980) [61].

This chapter deals with a simple supply chain that is modelled as system with a single warehouse, a distribution centre and single retailer (all retailers are identical in character), handling a single perishable product. In order to avoid the complexity, at the same time without loss of generality, we assumed the Poisson demand pattern at retailer node. This restricts our study to design and analyse as the tandem network of inventory, which is the building block for the whole supply chain system.

7.2 The Model Description

We consider a continuous review two-echelon perishable inventory system in supply chain implementing partial backlogging at retailer node. A finished product is supplied from distribution centre to retailer who adopts (s,S) policy for maintaining inventory. The demand at retailer node follows a Poisson process with mean rate $\lambda>0$. Supply to the retailer in pockets of Q=S-s items is administered with exponential lead time having parameter $\mu_0>0$. The replenishment of items in terms of pockets at distribution centre is made from a Ware house (WH) which apt one –for-one replenishment policy (M-1,M). The lead time for replenishment is exponentially distributed with parameter $\mu_1>0$ Demands occurring during the stock out periods are backlogged up to a specified quantity 'b' at retailer node(where b is the backorder level such that Q>b+s (that is Q-s>b>0). In this model, the maximum inventory levels M and S are fixed and reorder level is assumed to be s such that S-s=Q>b+s and M=nQ, $n\in N$. Each item in inventory has the perishable rate $\gamma(>0)$ with exponentially distributed life time.

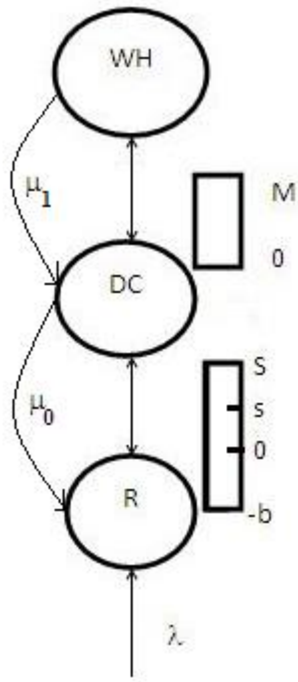


Figure1: Inventory Control with Partial Backlogging in Supply Chain

7.3 Analysis:

7.3.1 System Analysis

Let $I_0(t)$ and $I_1(t)$ denote the on hand inventory levels at retailer node and distribution centre respectively at time t. Then $I(t) = \{I_0(t), I_1(t) : t \geq 0\}$ is a Markov process with state space E=E$_1$×E $E_1 = \{S, (S-1), ..., s, (s-1), ..., 2, 1, 0, -1, -2, -3, ..., -b\}$, $E_2 = \{nQ, (n-1)Q, ..., Q, 0\}$,

where b is backorder level such that Q>b+s that is0≤b≤Q-s(we use negative sign for backlogging quantity. Assume the items perish at retailer node.

State transitions are given below:

(i) The arrival of a demand for an item at retailer node makes a state transition in the Markov Process from (j,q) to (j-1,q) with rate λ+jγ.

(ii) Replenishment of inventory at retailer node makes a state transition from (j,nQ) to (j+Q, (n-1)Q) with rate μ_0>0.

(iii) Replenishment of inventory at distribution centre node makes a state transition fro (j, (n-1)Q) to (j, nQ) with rate $\mu_1>0$.

(iv) The state (j,q) will transmit to (j-1,q) with perishing rate $j\gamma$.

7.3.2 MDP Formulation

Decision epochs: Random points of time at which a demand occurs for a single item at retailer node.

State Space: The state space E is partitioned as follows:

$E_1 = S, (S-1), ..., s, (s-1), ..., 2, 1, 0, -1, -2, -3, ..., -b$, $E_2 = nQ, (n-1)Q, ..., Q, 0$,

where b is backorder level such that Q>b+s that is$0 \le b \le Q-s$(we use negative sign for backlogging quantity).

Action Set: The reordering decisions (0- no order;1- order; 2 –compulsory order) taken at each state of the system (j, q) $\in$ E. The compulsory order for S items is made when inventory level is zero. Let A_i (i =1, 2, 3) denotes the set of possible actions. Where, $A_1 = \{0\}$, $A_2 = \{0, 1\}$, $A_3 = \{2\}$ and $A = A_1 \cup A_2 \cup A_3$.

The set of all possible actions are at $i \in E$.

$$A_i = \begin{cases} \{0\}, & s+1 \le j \le S \\ \{0,1\}, & 1 \le j \le s \\ \{2\}, & j = 0 \end{cases} , \quad A = \bigcup_{i \in E} A_i .$$

Transition Probability: $p_{(j,q)}^{(k,r)}(a)$ denote the transition probability from state (j, q) to state (k, r) when decision a is made at state (j, q).

$$R = \begin{array}{c} nQ \\ (\mathrm{n}-1)\mathrm{Q} \\ (\mathrm{n}-2)\mathrm{Q} \\ (\mathrm{n}-3)\mathrm{Q} \\ \vdots \\ Q \\ 0 \end{array} \begin{pmatrix} A & B & 0 & 0 & \cdots & 0 & 0 \\ C & D & B & 0 & \cdots & 0 & 0 \\ 0 & C & D & B & \cdots & 0 & 0 \\ 0 & 0 & C & D & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & D & B \\ 0 & 0 & 0 & 0 & \cdots & C & E \end{pmatrix}$$

The entries of the block partition matrix R can be written as

$$R_{q\times r} = \begin{cases} A \quad if \quad q = nQ & ; \quad r = q \\ B \quad if \quad q = nQ, (n-1)Q, (n-2)Q, ..., 2Q, Q; & r = q - Q \\ C \quad if \quad q = (n-1)Q, (n-2)Q, ..., 2Q, Q, 0 & ; \quad r = q + Q \\ D \quad if \quad q = (n-1)Q, (n-2)Q, ..., 2Q, Q & ; \quad r = Q \\ E \quad if \quad q = 0 & ; \quad r = q \\ 0 \quad otherwise \end{cases}$$

The sub matrices A, B, C, D and E are given by

$$A_{j\times q} = \begin{cases} \lambda + j\gamma & if \quad q = j-1, \quad j = S, S-1, S-2, ..., 2, 1. \\ \lambda & if \quad q = j-1, \quad j = 0, -1, -2, ..., -(b-1). \\ -(\lambda + j\gamma) & if \quad q = j, \quad j = S, S-1, S-2, ..., (s+1). \\ -(\lambda + j\gamma + \mu_0) & if \quad q = j, \quad j = s, s-1, s-2, ..., 2, 1. \\ -(\lambda + \mu_0) & if \quad q = j, \quad j = 0, -1, -2, ..., -(\mathrm{b}-1). \\ \mu_0 & if \quad q = j, \quad j = 0 \\ 0 & otherwise. \end{cases}$$

$$B_{j\times q} = \begin{cases} \mu_0 \quad if \quad q = j + Q, \quad j = s, s-1, s-2, ..., 2, 1, 0, -1, -2, ..., -b. \\ 0 \quad otherwise \end{cases}$$

$$C_{j\times q} = \begin{cases} \mu_1 \quad if \quad q = j \quad , \quad j = S, S-1, S-2, ..., s, s-1, s-2, ..., 2, 1, 0, -1, -2, ..., -b. \\ 0 \quad otherwise \end{cases}$$

$$D_{j\times q} = \begin{cases} \lambda + j\gamma & if \quad q = j-1, \quad j = S, S-1, S-2, \ldots, 2, 1. \\ \lambda & if \quad q = j-1, \quad j = 0, -1, -2, \ldots, -(b-1). \\ -(\lambda + j\gamma + \mu_1) & if \quad q = j, \quad j = S, S-1, S-2, \ldots, (s+1). \\ -(\lambda + j\gamma + \mu_0 + \mu_1) & if \quad q = j, \quad j = s, s-1, s-2, \ldots, 2, 1. \\ -(\lambda + \mu_0 + \mu_1) & if \quad q = j, \quad j = 0, -1, -2, \ldots, -(b-1). \\ -(\mu_0 + \mu_1) & if \quad q = j, \quad j = 0 \\ 0 & otherwise. \end{cases}$$

$$E_{j\times q} = \begin{cases} \lambda + j\gamma & if \quad q = j-1, \quad j = S, S-1, S-2, \ldots, 2, 1. \\ \lambda & if \quad q = j-1, \quad j = 0, -1, -2, \ldots, -(b-1). \\ -(\lambda + j\gamma + \mu_1) & if \quad q = j, \quad j = S, S-1, S-2, \ldots, (s+1). \\ -(\lambda + \mu_1) & if \quad q = j, \quad j = 0, -1, -2, \ldots, -(b-1). \\ -\mu_1 & if \quad q = j, \quad j = 0 \\ 0 & otherwise. \end{cases}$$

The steady state balance equations are

$$\upsilon^{nQ} A + \upsilon^{(n-1)Q} C = 0$$

$$\upsilon^{iQ} B + \upsilon^{(i-1)Q} D + \upsilon^{(i-2)Q} C = 0 \quad ; \text{i=n,n-1,n-2,…….2}$$

$$\upsilon^{Q} B + \upsilon^{0} E = 0$$

and the normalizing equation $\sum_{(j,q)\in E} \upsilon_j^q = 1$

Solve the above equations we get the steady state probability distribution υ_j^q

Cost function:

$c_{(j,q)}(a)$ denote the cost occurred in the system when action'a' is taken at state (j, q).

7.4 System Performance Measures

1.Mean Reorder Rates:

Consider the event β_i of reorders at node i (i=0, 1). Observe that β_0 denote the event that the inventory level at retailer node reaches s, whereas the β_1 an event that the inventory level at distribution centre reaches the level M-Q (i.e. n-1 packets). The mean reorder rate at retailer node is given by

$$\beta_0 = (\lambda + j\gamma)\sum_{j=0}^{s+1} \sum_{q=0}^{nQ}{}^{*} v_j^q$$

The mean reorder rate at the distribution centre is given by

$$\beta_0 = \mu_0 \sum_{j=0}^{s} \sum_{q=0}^{(n-1)Q}{}^{*} v_j^q$$

2. Mean Inventory Levels:

Let $\overline{I}_i$ denote the mean inventory level in the steady state at node i (i=0, 1).

Thus the mean inventory level at retailer node is given by

$$\overline{I}_0 = \sum_{q=0}^{nQ}{}^{*} \left(\sum_{j=0}^{S} j \cdot v_j^q \right)$$

The mean inventory level at distribution centre is given by

$$\overline{I}_1 = \sum_{j=0}^{S} \left(\sum_{q=0}^{nQ}{}^{*} q \cdot v_j^q \right).$$

3. Mean Shortage Rates:

Let α_i denote the shortage rate at node i (i=0, 1). Thus the shortage rate at retailer node is given by

$$\alpha_0 = \lambda \sum_{q=0}^{nQ}{}^{*} v_{-b}^q$$

The shortage rate at distribution centre α_1 is given by

$$\alpha_1 = (\lambda + j\gamma)\sum_{j=0}^{s+1} {}^{*} v_j^0.$$

4. Mean Perishing Rates:

$$\alpha_2 = \sum_{q=0}^{nQ} \sum_{j=0}^{S} j\gamma \; v_j^q$$

Cost Analysis:

The long-run expected cost rate when policy f is adopted is given by

$$C(s,Q) = h_0\,\overline{I}_0 + h_1\overline{I}_1 + k_0\beta_0 + k_1\beta_1 + g_0\alpha_0 + g_1\alpha_1 + g_2\alpha_2$$

where k_0 and k_1 are setup costs (ordering cost) regardless of order size; h_0 and h_1 be holding cost per unit item per unit time at retailer and distributor nodes respectively; g_i is the shortage cost at node i (i=0, 1) for unit shortage per unit time and g_2 is the perishing cost per unit item. Hence the average cost rate of the system is given by

$$\begin{aligned}\mathrm{C}(s,Q) = {} & h_0\left(\sum_{a\in\{0,1,2\}}\sum_{q=0}^{nQ}{}^{*}\left(\sum_{j=0}^{S} j\cdot v_j^q(a)\right)\right) + h_1\left(\sum_{a\in\{0,1,2\}}\sum_{j=0}^{S}\left(\sum_{q=0}^{nQ}{}^{*} q\cdot v_j^q(a)\right)\right) + k_0\left(\sum_{a\in\{0,1,2\}}\lambda\sum_{q=0}^{nQ}{}^{*} v_{s+1}^q(a)\right) \\ & + k_1\left(\sum_{a\in\{0,1,2\}}\mu_0\sum_{j=0}^{s}\sum_{q=0}^{(n-1)Q}{}^{*} v_j^q(a)\right) + g_0\left(\sum_{a\in\{0,1,2\}}\lambda\sum_{q=0}^{nQ}{}^{*} v_{-b}^q(a)\right) + g_1\left(\sum_{a\in\{0,1,2\}}\mu_0\sum_{q=0}^{nQ} v_j^0(a)\right) \\ & + g_2\left(\sum_{q=0}^{nQ}\sum_{j=0}^{S} j\gamma\ v_j^q)\right)\end{aligned}$$

When policy R is adopted. From our assumptions, it can be seen that the controlled process $(I_0{}^R, I_1{}^R)$ is a finite state semi-Markov decision process. A policy R is called a stationary policy if it is randomized, time invariant and Markovian. Further, a process is said to be completely ergodic if every stationary policy give rise to an irreducible Markov chain. From our assumptions it can be seen that for every stationary policy f, $(I_0{}^f, I_1{}^f)$ is completely ergodic. Since the action space is also finite, a stationary optimal policy exists. Hence we consider the class f of all stationary policies.

Our objective is to find an optimal policy f^* for which $C^{f^*} \le C^f$ for every f.

For any fixed $f \in F$ and $(\mathrm{j},q),(\mathrm{k},r) \in E,$ define

$$P_{jq}^f(\mathrm{k},r,t) = \Pr[I_0{}^f(t) = k, I_1{}^f(t) = r \mid I_0{}^f(0) = j, I_1{}^f(0) = q],\ (\mathrm{j},q),(\mathrm{k},r) \in E$$

Then $P_{jq}(\mathrm{k},r,t)$ satisfies the Kolmogorov forward differential equations. As each policy, f, results in an irreducible Markov chain and action spaces are finite, $P^{f}(\mathrm{k},r)=\lim P_{jq}(\mathrm{k},r,t)$ exists and is independent of the initial conditions. The condition $\sum_{(\mathrm{j,k})\in E} P^{f}(\mathrm{j,k})=1$, determine the steady-state probabilities uniquely

7.5 Linear programming problem

7.5.1 LPP Formulation

Let us define the variables D(j, q, a) as

D(j, q, a)= Pr [decision is a \ state is (j, q)].

Then for any stationary policy f, we have D(j, q, a) =0 or 1.Suppose D(j, q, a) were continuous variable (instead of integers), then the semi-Markov decision problem can be reformulated as a linear programming problem . For this purpose we consider the class of all randomized , time –invariant Markovian policies for which the probability functions D(j, r, k) satisfy

$$0 \leq D(j,q,a) \leq 1$$

and

$$\sum_{a\epsilon A} D(j,q,a) = 1, 0 \leq j \leq S, 0 \leq q \leq M$$

The linear programming problem is best expressed in terms of the variables y(j, q, a), which are defined as

$y(j, q, a) = D(j, q, a)P^{f}(j,q)$, (1)

As y(j, q, a)=Pr[state is (j ,a) and decision is a],for any given f, we have

$p^{f}(j,q)= \sum_{a\epsilon A} y(j,q,a) \quad (j,q)\epsilon E$ (2)

Expressing $P^{f}(j, q)$ in terms of y(j, q, a) , we obtain the following linear programming problem:

Minimize

$$C(s,Q)=h_0\left(\sum_{a\in\{0,1,2\}}\sum_{q=0}^{nQ}{}^{*}\left(\sum_{j=0}^{S} j\cdot v_j^q(a)\right)\right)+h_1\left(\sum_{a\in\{0,1,2\}}\sum_{j=0}^{S}\left(\sum_{q=0}^{nQ}{}^{*} q\cdot v_j^q(a)\right)\right)+k_0\left(\sum_{a\in\{0,1,2\}}\lambda\sum_{q=0}^{nQ}{}^{*} v_{s+1}^q(a)\right)$$

$$+k_1\left(\sum_{a\in\{0,1,2\}}\mu_0\sum_{j=0}^{s}\sum_{q=0}^{(n-1)Q}{}^{*} v_j^q(a)\right)+g_0\left(\sum_{a\in\{0,1,2\}}\lambda\sum_{q=0}^{nQ}{}^{*} v_{-b}^q(a)\right)+g_1\left(\sum_{a\in\{0,1,2\}}\mu_0\sum_{q=0}^{nQ} v_j^0(a)\right)$$

$$+g_2\left(\sum_{q=0}^{nQ}\sum_{j=0}^{S} j\gamma\ v_j^q)\right)$$

The constraints of the linear programming problem are as follows:

a) $y(j,q,a)\geq 0$ (j,q)$\in E, a\in A_i$, i=0,1,2. (3)

b) $\sum_{(j,q)\in E} P^f(j,q)=1$ and

$$\sum_{i=0}^{2}\sum_{(j,q)\in E_l}\sum_{k\in A_l} y(j,r,k)=1$$

c) $$\sum_{i=0}^{2}\sum_{(j,q)\in E}\sum_{a\in A_i} y(j,q,a)=1,$$ (4)

d) The remaining constraints are the balance equations

As we can see from the lemma below solving the linear programming problem gives the optimal solution when the $y(j,r,k)$ are constrained to be integers.

The optimal solution of the above linear programming problem yields a deterministic policy.

From equations (1) and (2), we have

$$D(j, r, k)=\frac{y(j,M,k)}{\sum_{k=0}^{K} y(j,r,k)} \quad (5)$$

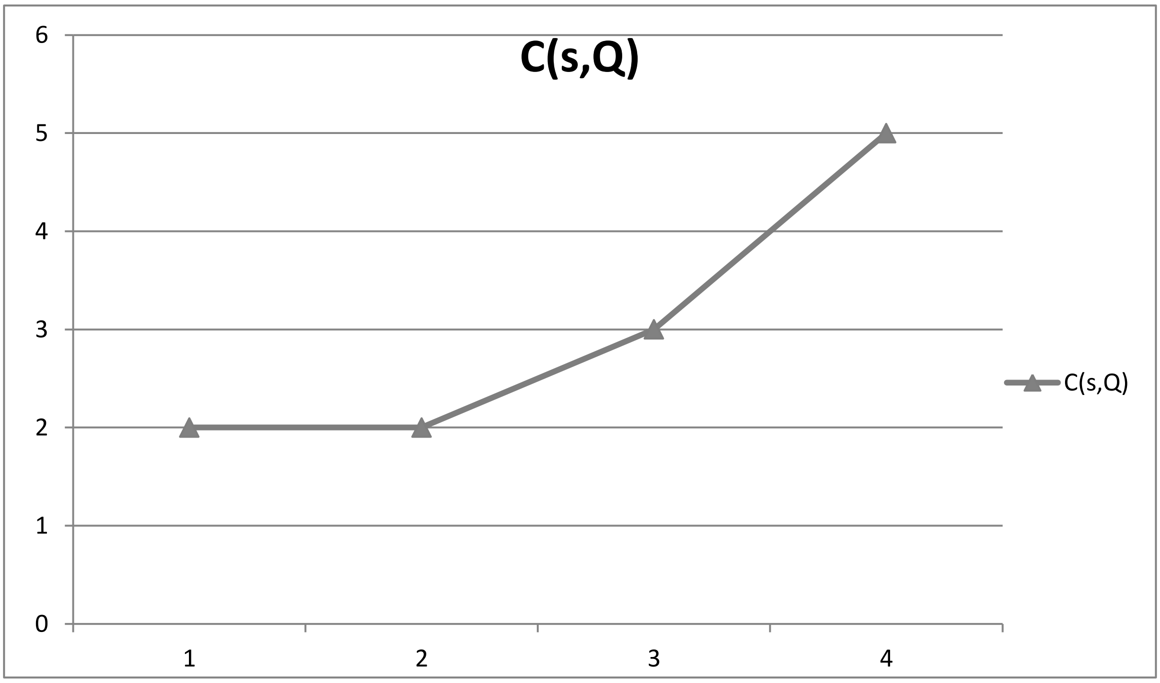
Since the decision problem is completely ergodic, every basic feasible solution to the above linear programming problem has the property that for each(j,r)$\in E$,D(j,r,k) is 1 for exactly one value of k and zero for all other values of k. Thus ,given the amount of inventory on-hand and the number of customers in the system ,we have to choose the service rate γ_k for which D(j,r,k) is1.Hence any basic feasible solution of the linear programming yields a deterministic policy.

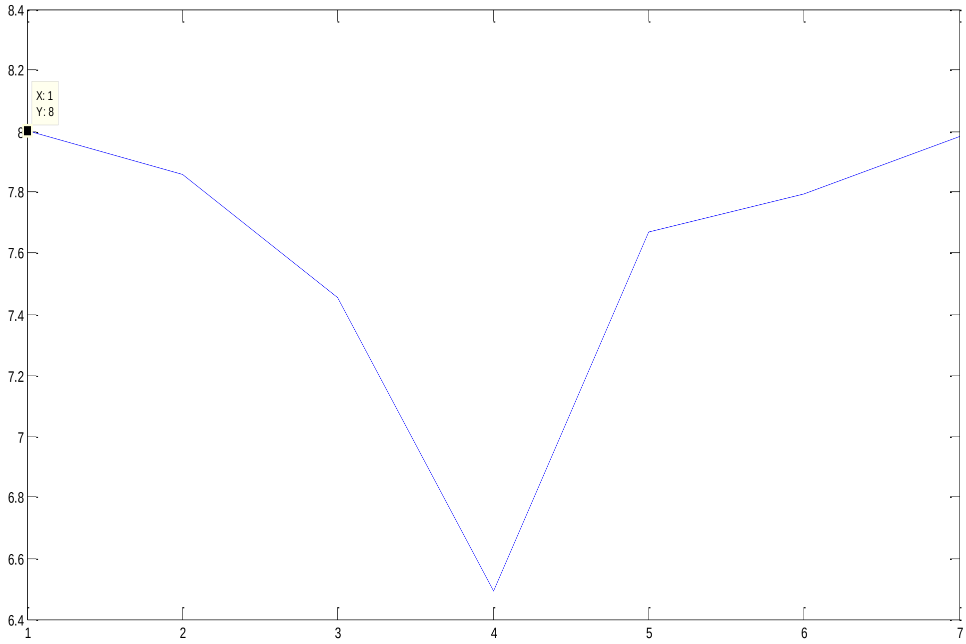
7.6 Numerical illustration and discussion:

Total excepted cost rate C(s, Q) as a function s and Q is given in the following table

s	Q	C(s,Q)
0	7	8.0002
1	6	7.8571
2	5	7.4543
*3	*4	*6.4932
4	3	7.6702
5	2	7.7945
6	1	7.9820

The graphical representation of the long run expected cost rate C(s*,Q*) is given below:





7.7 Conclusion

Analysis of inventory control in Supply Chain has been obtained by many researchers in the recent past. But most of them are in the direction of determining. The optimal inventory level when a fixed policy is implemented. We, in this article studied the problem in a new approach for a perishable inventory in Supply Chain . For a given Supply Chain structure (two-stage), the ordering quantities are controlled with MDP, and the optimal ordering policy is founded. For both DC and Retailer only partial backlogging is considered and perishable is only at retailer node for study, one may extend this full backlogging in future.

ANNEXURE – I

LIST OF PUBLICATIONS

1. ***P.K. Santhi, Gowsalya, C.Elango,*** *Perishable Inventory Control in Supply Chain : A Semi-MDP Model in the International Journal of Computational and Applied Mathematics, Volume 12,Number 1, ISSN No. :1819-4966, (2017)*
2. ***P.K. Santhi, C.Elango,*** *MDP in supply chain: Inventory system with service facility at retail node with impatient customer" in Journal of Management Science and Humanities, Vol.4(2), 132-144 ,December 2017,ISSN No:2395 -0625,(2017)*
3. ***P.K. Santhi, C.Elango,*** *Optimal Inventory Control with Partial Backlogging in Supply Chain: MDP Approach " in the International Journal of Research in Information Technology, Vol.2, Issue2, Feb 2018, Pg 30-31, ISSN(online):2001-5569*
4. ***P.K. Santhi, C.Elango****, Optimal Inventory Control For Perishable Inventory with partial Backlogging in a Supply Chain: Markov Decision Process " in the International Journal of Fuzzy Mathematical Archive , Vol.15, No.2,2018, 30th April 2018, PP No : 167-176,ISSN: 2320-3242(P),2320-3250(online).*
5. ***P.K.Santhi, C.Elango ,****MDP in Supply Chain: Optimal Inventory Control System " in the International Journal of Engineering Science Invention (IJESI), ISSN(online):2319-6734, ISSN (print) : 2319-6726,Vol.7, Issue 8,Version II, 22nd August 2018, PP No : 55-62,Impact Factor:5.962, UGC(Approved).*

LIST OF PRESENTATIONS

1. *Paper Presented, entitled "MDP in Supply Chain Inventory Control System" in the International Conference on Mathematical Modeling and Computational Methods in Science and Engineering (ICMMCMSE- 2017) jointly organized by Ramanujan Centre for Higher Mathematics and Department of Mathematics ,Alagappa University, Karaikudi, (20th -22nd February, 2017).*

2. *Paper Presented, entitled" Perishable Inventory Control System in Supply Chain: A semi-MDP Model" in the UGC Sponsored National conference on Mathematical Modelling(NCOMM-2017) organized by PG & Research Department of Mathematics, Cardamom Planter's Association College, Bodinayakanur, Theni District,TamilNadu(30th -31st March,2017)*

3. *Paper Presented, entitled "MDP in supply chain Inventory system with service facility at retail node with impatient customers" in the two days National Conference on Recent Advancements in Pure and Applied Mathematics organised by Department of Mathematics ,Nadar Saraswathi College of Arts and Science, Vadaputhupatti Theni D.t (26th - 27th July ,2017)*

4. *Paper Presented, entitled "Inventory Control in a service Facility System is Supply Chain:Semi Markov Decision Process" in the National Seminar on Recent Trentsin Applied Mathematics organised by the Research Centre & PG Department of Mathematics ,Jayaraj Annapackiam College for Women(Autonomous), Periyakulam, Theni D.t (13th December ,2017)*

5. *Paper Presented, entitled "Optimal Inventory Control with Partial Backlogging in a Supply Chain: Markov Decision Process" in the International Conference on Recent Trends in Stochastic Modeling and its Applications (ICRTSMA 2018) organized by Department of Statistics ,Manonmaniam Sundaranar University, Tirunelveli, (8th -9th , January, 2018).*

6. *Paper Presented, entitled "Optimal Control for Perishable inventory with PartialBacklogging in a Supply Chain:MDP approach" in the one day International Seminar on Algebra and Applied Mathematics organised by Department of Mathematics ,Hajee Karutha Rowther Howdia College, Uthamapalayam,Theni D.t(11thJanuary,2018*)
7. *Paper Presented, entitled "Optimal Control of Supply Chain with Service Facility having Impatient Customers" in the TNSCST sponsored International Conference on Emerging Trends in Mathematical Sciences and Technology organised by the Research Centre & PG Department of Mathematics ,Jayaraj Annapackiam College for Women(Autonomous), Periyakulam, Theni D.t (20th -21st December ,2018*)

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